1. Use the definition of matrix multiplication to show that the product of two upper triangular matrices is upper triangular. Can you adopt your proof for lower triangular matrices?

Solution. Let U, V be upper triangular with entries u_{ij} and v_{ij}. It suffices to show that the $ij$the entry of UV is 0 when $i > j$, i.e. $(UV)_{ij}$ is below the diagonal. By definition, we know that

$$(UV)_{ij} = \sum_{k=1}^{n} u_{ik} v_{kj}.$$

Since U and V are upper triangular, we know that $u_{ik} = 0$ when $i > k$ and $v_{kj} = 0$ when $k > j$. But for each $k \in \{1, \ldots, n\}$, one of these inequalities is true since $i > j$. (In fact both inequalities are true when k is in between i and j.) Therefore either $u_{ik} = 0$ or $v_{kj} = 0$ for all k from 1 to n. Thus when $i > j$,

$$(UV)_{ij} = \sum_{k=1}^{n} u_{ik} v_{kj} = \sum_{k=1}^{n} 0 = 0$$

This means that UV is upper triangular as desired. The same proof works when two matrices L and M are lower triangular. Simply change $i < j$.

2. (1.2.25) Find all solutions to the matrix equation $AX = I_2$ where $A = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$ using row reduction. Need X be 2×2?

Solution. Let $X = \begin{pmatrix} x & u \\ y & v \end{pmatrix}$. By the column vector definition of matrix multiplication, we know the columns of AX are $A\begin{pmatrix} x \\ y \end{pmatrix}$ and $A\begin{pmatrix} u \\ v \end{pmatrix}$. Therefore to find X, it suffices to find the solutions to the two systems of equations

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad A\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Instead of doing row reduction on the same matrix twice, we can make our augmented matrix larger and row reduce

$$\left(\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right)$$

to get the identity on the left, and the solution X will be on the right. After doing the row operations swap(r_1, r_2), $r_2' = -2r_1 + r_2$, $r_1' = -r_2 + r_1$, the reduced matrix is

$$\left(\begin{array}{cc|cc} 1 & 0 & -1 & 3 \\ 0 & 1 & 1 & -2 \end{array}\right)$$

so that

$$X = \begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix}.$$

The matrix X needs to be 2×2. If X is $n \times m$, then $n = 2$ since AX is well defined. But since AX is 2×2, this requires $m = 2$ as well.

3. Let the commutator $[A, B]$ be defined by

$$[A, B] = AB - BA.$$

We know that matrix multiplication does not commute, so the commutator is not necessarily zero. Show that the commutator satisfies the Jacobi relation

$$[[A, B], C] + [[C, A], B] + [[B, C], A] = 0.$$

Solution. This problem can be solved by expanding the commutators in the Jacobi identity, and noting that every term cancels.

$$
\begin{aligned}
[A,B],C] + [[C,A],B] + [[B,C],A] &= (AB - BA)C - C(AB - BA) \\
&\quad + (CA - AC)B - B(CA - AC) + (BC - CB)A - A(BC - CB) \\
&= ABC - BAC - CAB + CBA + CAB \\
&\quad - ACB - BCA + BAC + BCA - CBA - ABC + ACB \\
&= 0
\end{aligned}
$$