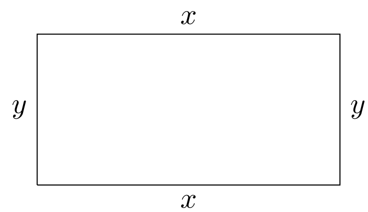
Problem 1. Remember from the word problems on the exams that we retired to Cocoyashi village to start a tangerine farm. In our imaginary scenario, we have 40 meters of fencing to make a rectangular boundary for our tangerine grove. Let's call the width x, the length y, and the area A. Since we have 40 meters of fencing, the perimeter will have to be 40 meters.



$$P = 2x + 2y = 40$$
$$A = xy$$

Remember the perimeter of a rectangle is the sum of the sides, so in this case Perimeter $= x + y + x + y = 2x + 2y$. But we only have 40 meters of fencing, so it must be that $2x + 2y = 40$.

a) Use the equation $2x + 2y = 40$ to solve for the length y in terms of the width x. Since we have a fixed perimeter, they depend on each other.

Now remember, that the area of a rectangle is the width times the length. In our variables, $A = xy$. But we wrote y in terms of x already!

b) Use your answer from part a) to find the area A in terms of the width x when the perimeter is 40. You should get the parabola $A = 20x - x^2$. Can you explain why this formula represents a parabola?

Now we can use what we know about parabolas. Here the width x will be the independent variable, and A will be the dependent variable (or the y-axis).

c) Does this parabola face up or face down?

d) Find the vertex of the parabola $A = 20x - x^2$. In terms of the word problem, which rectangle does it represent? Do all x values give you an actual rectangle?

e) Find the roots of the parabola $A = 20x - x^2$. Which "rectangles" do the roots represent in terms of our word problem?

Problem 2. Suppose a swarm of tangerine tree aphids has descended on our grove. (An aphid is a bug.) The aphids secrete a bacteria which infects the tangerines and makes them taste bad! The aphids got on all 15000 tangerines, but there's an 35% chance tangerines becomes infected. Suppose we find a test which can tell whether the tangerines are infected with aphid bacteria or not. For infected tangerines, it has a 99% success rate, but for uninfected tangerines, it has a 95% success rate.

a) Given that we have 15000 tangerines, and each has a 35% chance of getting infected, figure out why the expected value of infected tangerines is 5250.

Nothing is guaranteed in probability problems, but for fun let's assume we've hit the expected value of tangerines. So use 5250 for the number of infected tangerines for the next two parts.

b) Suppose that we have 15000 tangerines total, and 5250 infected tangerines. If a tangerine is tested to have an aphid bacteria infection, what is the probability that it actually has one?

c) What about the opposite situation; if a tangerine tests negative for aphid bacteria infection, what is the probability that it is uninfected?

3. Let's define the Fibonacci sequence f_n. This will be a sequence of integers, defined by the following recursive equation.

$$f_n = f_{n-1} + f_{n-2} \quad f_0 = 1 \quad f_1 = 1$$

You start off with 1,1, and then add the previous two numbers to get the next one. The sequence starts off $1, 1, 2, 3, 5, 8, 13, \ldots$. Notice that $1 + 1 = 2$, $1 + 2 = 3$, $2 + 3 = 5$, etc.

a) In a spreadsheet, calculate the first 50 terms of the sequence, i.e. the numbers $f_0, f_1, \ldots, f_{50}$. It'll start the get big pretty fast!

b) Now in your spreadsheet, calculate all the ratios of successive terms. Explicitly, f_1/f_0, then f_2/f_1, f_3/f_2..., f_{50}/f_{49}. Your answers should all start being around 1.618... eventually. It is actually the famous "golden ratio" if you are familiar with the term!

c) We can actually predict why we should get 1.618.... Assume that for really large terms, your ratios are all about the same. We are going to pick some large numbers just for an example. We'll go with 48, 49, and 50. So for example

$$\frac{f_{50}}{f_{49}} \approx \frac{f_{49}}{f_{48}}.$$

Let's call this ratio $x = f_{50}/f_{49} \approx f_{49}/f_{48}$, and we'll calculate why it is 1.618....

I'll give you some directions for what algebra to do! Write down the recursive formula for $n = 50$.

$$f_{50} = f_{49} + f_{48}$$

Now divide both sides by f_{49} and plug in your variable $x = f_{50}/f_{49}$ and $x \approx f_{49}/f_{48}$ to get

$$x \approx 1 + \frac{1}{x}.$$

Multiply x by both sides, and move all the terms to one side to get

$$x^2 - x - 1 \approx 0.$$

We can ignore the approximation and make it an equals

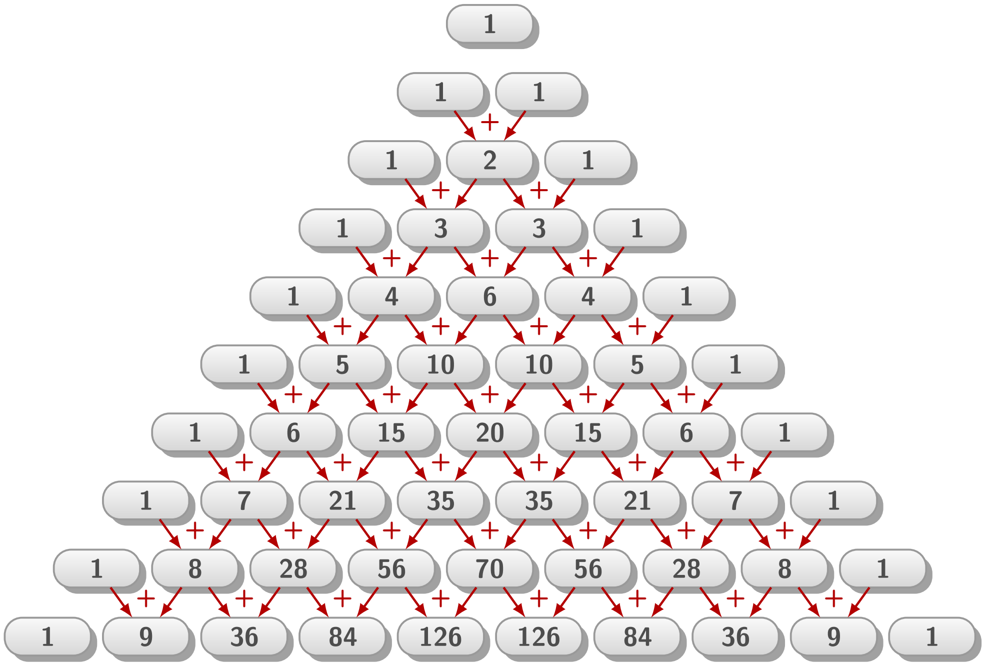
$$x^2 - x - 1 = 0.$$

This is a quadratic equation, but it doesn't factor with integers unfortunately. But we can find a solution by graphing the corresponding parabola.

$$y = x^2 - x - 1$$

Graph $y = x^2 - x - 1$ using a graphing calculator and see that one of the roots is 1.618! Remember that a root is when the parabola intersects the x-axis.

4. Let's recall how Pascal's triangle worked. We started off with 3 1's in a triangle in the top, and then got new rows by adding 1's on the sides, and adding the top left and top right numbers together to get the next row.



(This picture was adapted from here: https://texample.net/tikz/examples/pascals-triangle-and-sierpinski-triangle/)

We can use our understanding of probability to figure out why this adding property works! Remember the set up: suppose we have n coins, and we want to find the number of ways we can get k heads. If $n = 4$ and $k = 2$, we have 4 coins and we want to find the number of ways to get 2 heads. We could count them out...

$$\begin{array}{ccc} HHTT & HTHT & HTTH \\ THTH & THHT & TTHH \end{array}$$

and say "Hey, we wrote out 6 ways, so the answer is 6!" Or we could go to the 4th row and 2nd position on the triangle, and read off the number. It is 6.

In this problem, we will understand why Pascal's triangle can be figured out from the "adding top left and top right" property.

a) Let's make the numbers bigger. Suppose we are flipping 9 coins and we want to find the number of ways to get 4 heads. Which number on Pascal's triangle tells you the answer.

b) Imagine we flip the first of the 9 coins and the first one comes up tails. Which number on Pascal's triangle represents the number of ways we can get 4 heads in the remaining 8 coins?

c) Now imagine that we flip the first of the 9 coins and the first one comes up heads! Well we already got one of the heads we wanted, so we have 8 coins and 3 heads to go. Which number on Pascal's triangle represents the number of ways to get 3 heads out of 8 coins.

d) Now we can solve the problem. Notice that your answers to b) and c) add up to your answer to a). Why? (Hint: The first coin of 9 is either heads or tails, so...) What does this have to do with our adding property in Pascal's triangle?

e) Did our analysis have anything to do with $n = 9$ and $k = 4$? What if we changed the numbers to $n = 7$ and $k = 5$? Would the relationships between the numbers of Pascal's triangle change?