Topics

- $m \times n$ matrix
- ij^{th} entry in a matrix A, a_{ij}
- row vector vs. column vector
- zero matrix
- identity matrix
- diagonal matrix
- elementary matrix
- upper/lower triangular matrix
- regular matrix
- LU decomposition
- nonsingular matrix
- permutation matrix
- permuted LU decomposition
- matrix inverse
- matrix transpose
- Theorem 1.18
- Lemma 1.19 - Lemma 1.21
- Theorem 1.28
- LDV decomposition
- Lemma 1.32
- symmetric matrix
- Theorem 1.34
- row-echelon form
- pivots
- matrix rank
- basic variables, free variables
- Theorem 1.45
- determinant of a matrix
- Theorem 1.50
- Lemma 1.51
- Theorem 1.52
- vector spaces
- $\mathbb{R}^n$, polynomials, $C^0(\mathbb{R})$
- subspaces
- linear combination
- span
- linearly independent
- linearly dependent
- Theoerem 2.21
- rank of a matrix
- basis
- Theorem 2.29
- Theorem 2.31
- Lemma 2.34
- kernel
- image
- superposition principle
- Proposition 2.41

1. Find all 2×2 matrices A which satisfy the equation

$$A^2 = 2I.$$

2. Compute the permuted LDV decomposition of the matrix

$$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 3 \\ 1 & -1 & 0 \end{pmatrix}.$$

Determine the rank and dimension of the kernel as well.

3. Let P^n be the vector space of polynomials of degree $\leq n$. Find the dimension of this vector space.

4. (a) Let $A = \begin{pmatrix} -1 & 3 \\ 1 & 2 \end{pmatrix}$ and $x = \begin{pmatrix} x \\ y \end{pmatrix}$. Compute the expression $x^T A x$. (b) Consider the polynomial in two variables $2x^2 + xy + 3y^2$. Write find a matrix B such that polynomial in the form

$$2x^2 + xy + 3y^2 = x^T B x.$$

(c) Show that every polynomial of the form $ax^2 + bxy + cy^2$ can be written in the form $x^T M x$ where M is a symmetric matrix.

5. Let V be a vector space. Let U and W be subspaces of V. (a) Show that the intersection $U \cap W$ is a subspace. (b) Let $V = \mathbb{R}^4$. Find subspaces U and W such that $\dim U \cap W = 0$.

6. Recall that the trace of a square matrix A is the sum of its diagonal elements.

$$\operatorname{tr} A = \sum_i a_{i,i} = a_{1,1} + \cdots + a_{n,n}$$

Show that the set of matrices A with $\operatorname{tr} A = 0$ is a subspace of the vector space $M_{n \times n}(\mathbb{R})$.

7. Determine whether the vector $\begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}^T$ is in the span of the vectors

$$\begin{pmatrix} -1 \\ 1 \\ -2 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ -1 \\ 2 \end{pmatrix} \quad \begin{pmatrix} 7 \\ -3 \\ 0 \\ 2 \end{pmatrix}.$$

8. Consider the vector space $C^0(\mathbb{R})$ of continuous functions on $\mathbb{R}$. Show that the functions $f(x) = \cos(2x)$, $g(x) = \cos^2(x)$ and $h(x) = 1$ are linearly dependent in this vector space.

9. (a) Find the numbers a such that the columns of the following matrix form a basis of $\mathbb{R}^3$.

$$A = \begin{pmatrix} a & 1 & 2 \\ 0 & a & 1 \\ -1 & 2 & a \end{pmatrix}$$

(b) For what a is the rank $A = 1$? How about rank $A = 2$?

10. Define $\mathbb{R}^\infty$ to be the set of all infinite sequences of real numbers. (a) Show that the set of all convergent subsequences is a subspace. (b) Determine whether C is finite or infinite dimensional.