1. Find all 2×2 matrices A which satisfy the equation

$$A^2 = 2I.$$

Solution. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then the above equations becomes

$$\begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Then $a, d = \pm\sqrt{2 - bc}$ and either $b, c = 0$ or $a = -d$. If $b, c = 0$, then the matrix has the form

$$\begin{pmatrix} \pm\sqrt{2} & 0 \\ 0 & \pm\sqrt{2} \end{pmatrix}.$$

Else if $a = -d$, then $bc \leq 2$ to keep the matrix real, so that the matrix has the form

$$\begin{pmatrix} \pm\sqrt{2 - bc} & b \\ c & \mp\sqrt{2 - bc} \end{pmatrix}.$$

2. Compute the permuted LDV decomposition of the matrix

$$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 3 \\ 1 & -1 & 0 \end{pmatrix}.$$

Determine the rank and dimension of the kernel as well.

Solution.

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 3 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 3/2 \\ 0 & 0 & 1 \end{pmatrix}$$

The matrix is invertible since it is nonsingular (there are 3 pivots). Thus the rank is 3 and the dimension of the kernel is 0.

3. Let P^n be the vector space of polynomials of degree $\leq n$. Find the dimension of this vector space.

Solution. The dimension of P^n is $n + 1$. We can show that the polynomials $\{1, x, \ldots, x^n\}$ form a basis of P^n. These functions are independent since if $c_0 + c_1 x + \ldots c_n x^n = 0$ as a function, then $c_i = 0$ since polynomials only have finite amount of roots. These functions span by definition of degree. Thus they form a basis and $\dim P^n = n + 1$.

4. (a) Let $A = \begin{pmatrix} -1 & 3 \\ 1 & 2 \end{pmatrix}$ and $x = \begin{pmatrix} x \\ y \end{pmatrix}$. Compute the expression $x^T A x$. (b) Consider the polynomial in two variables $2x^2 + xy + 3y^2$. Write find a matrix B such that polynomial in the form

$$2x^2 + xy + 3y^2 = x^T B x.$$

(c) Show that every polynomial of the form $ax^2 + bxy + cy^2$ can be written in the form $x^T M x$ where M is a symmetric matrix.

Solution. (a) $-x^2 + 4xy + 2y^2$ (b) $B = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$ (c) The matrix M has the form

$$\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}.$$

5. Let V be a vector space. Let U and W be subspaces of V. (a) Show that the intersection $U \cap W$ is a subspace. (b) Let $V = \mathbb{R}^4$. Find subspaces U and W such that $\dim U \cap W = 0$.

Solution. (a) Let $v, w \in U \cap W$. In particular both v and w are in both U and W. First $U \cap W$ is nonempty since $0 \in U$ and $0 \in W$. Then $v + w \in U$ since U is a subspace and similarly for W. Thus $v + w \in U \cap W$. Finally, if $c \in \mathbb{R}$, then $cv \in U$ since U is a subspace and similarly for W. Thus $cv \in U \cap W$.

(b) If we let $U = \operatorname{span}\{(1,0,0,0)^T, (0,1,0,0)^T\}$ and $W = \operatorname{span}\{(0,0,1,0)^T, (0,0,0,1)^T\}$. These subspaces have trivial intersection since all of the vectors are linearly independent.

6. Recall that the trace of a square matrix A is the sum of its diagonal elements.

$$\operatorname{tr} A = \sum_i a_{i,i} = a_{1,1} + \cdots + a_{n,n}$$

Show that the set of matrices A with $\operatorname{tr} A = 0$ is a subspace of the vector space $M_{n \times n}(\mathbb{R})$.

Solution. This subset is nonempty since the 0 matrix has zero trace. Let A and B be matrices with $\operatorname{tr}(A) = 0 = \operatorname{tr}(B)$. For additive closure, note that $\operatorname{tr}(A + B) = \operatorname{tr}(A) + \operatorname{tr}(B)$, since matrices add component wise. Therefore $\operatorname{tr}(A + B) = 0 + 0 = 0$. To show that scalar multiplication is closed, note that $\operatorname{tr}(cA) = \sum_i c(A)_{i,i} = c \operatorname{tr}(A) = 0$. Thus the trace zero matrices form a subspace.

7. Determine whether the vector $\begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}^T$ is in the span of the vectors

$$\begin{pmatrix} -1 \\ 1 \\ -2 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ -1 \\ 2 \end{pmatrix} \quad \begin{pmatrix} 7 \\ -3 \\ 0 \\ 2 \end{pmatrix}.$$

Solution. Put the three vectors in the first three columns of a matrix, and make $(1,2,3,4)^T$ the last column of the matrix. If the system has a free column in the fourth column, then the last vector is dependent on the other three. If it doesn't, then the fourth vector is independent of the other three.

This matrix actually row reduces to the identity.

$$\begin{pmatrix} -1 & 0 & 7 & 1 \\ 1 & 0 & -3 & 2 \\ -2 & -1 & 0 & 3 \\ 0 & 2 & 2 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Therefore the vector $(1,2,3,4)^T$ is not in the span of the other 3.

8. Consider the vector space $C^0(\mathbb{R})$ of continuous functions on $\mathbb{R}$. Show that the functions $f(x) = \cos(2x)$, $g(x) = \cos^2(x)$ and $h(x) = 1$ are linearly dependent in this vector space.

Solution. The double angle formula is

$$\cos(2x) = 2\cos^2(x) - 1.$$

Rewritten, this is a linear relationship between the functions so that $f(x) - 2g(x) + h(x) = 0$ and they are dependent.

9. (a) Find the numbers a such that the columns of the following matrix form a basis of $\mathbb{R}^3$.

$$A = \begin{pmatrix} a & 1 & 2 \\ 0 & a & 1 \\ -1 & 2 & a \end{pmatrix}$$

(b) For what a is the rank $A = 1$? How about rank $A = 2$?

Solution. (a) By the main theorem for square matrices, $\det A = 0$ iff the columns of A do not form a basis. Taking the determinant, we obtain that $\det A = a^3 - 1 = 0$. The only solution in the reals is $a = 1$. So when $a \neq 1$, the columns form a basis of $\mathbb{R}^3$.

(b) We know that rank $A = 3$ when $a \neq 1$. So we just have to check what the rank is when $a = 1$. In this case, the matrix row reduces to

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

which is rank 2 since it has 2 pivots. Alternatively, you could have seen that the first two columns of A are independent one is not a multiple of the other. Either way, the rank is 1 for no $a \in \mathbb{R}$, and for $a = 1$, the rank is 2.

10. Define $\mathbb{R}^\infty$ to be the set of all infinite sequences of real numbers. (a) Show that the set of all convergent subsequences is a subspace. (b) Determine whether C is finite or infinite dimensional.

Solution. First of all, the way I phrased the question doesn't tell you how $\mathbb{R}^\infty$ is a vector space. We can write a sequence as a tuple that just never ends.

$$(a_1, a_2, \dots)$$

You can add these like vectors in $\mathbb{R}^n$. They add component-wise, and scalar multiply component-wise as well. These operations satisfy the 7 axioms.

(a) Let $(a_i) = (a_1, a_2, \dots)$ and $(b_i) = (b_1, b_2, \dots)$ be convergent sequences. Since they're convergent, let $(a_i) \to a$ and $(b_i) \to b$. Recall that the sum of two convergent sequences is also convergent, so that $(a_i + b_i) \to a + b$. Thus C is closed under addition.

Given a scalar c, it is clear that $c(a_i) = (ca_i) \to ca$. Therefore C is closed under scalar multiplication as well. The set C is also nonempty (since $(0, 0, \dots) \in C$), and therefore C is a subspace of $\mathbb{R}^\infty$.

(b) This subspace is infinite dimensional. Assume for contradiction that there exists a finite basis $\{(x_i)_1, (x_i)_2, \dots, (x_i)_n\}$, so that $\dim C = n$. Then by Theorem 2.31, any set of sequences $\{(y_i)_1, \dots, (y_i)_k\}$ is linearly dependent when $k > n$. We can show that this leads to a contradiction by finding k linearly independent convergent sequences for $k > n$.

Pick any number $k > n$. Let (e^i_j) be the sequence defined by

$$\begin{cases} e^i_j = 0 & i \neq j \\ e^i_i = 1 \end{cases}$$

Here i is not an exponent. It is in index, I'm just putting where the exponent usually goes because there was already another index in the subscript. For example

$$(e^1_j) = (1, 0, 0, \dots) \quad \text{and} \quad (e^3_j) = (0, 0, 1, 0, 0, \dots).$$

These are essentially the standard basis vectors, but now they are sequences instead.

First, note that $(e^j_i) \to 0$ for all j. This is true since if we let $N > j$, then for all $n > N$, $|e^j_n - 0| = 0 < \epsilon$ for all $\epsilon > 0$. (Oops I used n twice, different n here.) So all of our "standard basis sequences" are convergent to zero, and $(e^j_i) \in C$.

Now consider the set of sequences

$$\{(e^1_i), (e^2_i), \dots, (e^k_i)\}$$

where $k > \dim C = n$ as you recall. By Lemma 2.31, this set of vectors should be dependent, since $k > \dim C$. But we can show that they are independent. For given a linear combination

$$c_1(e^1_i) + \cdots + c_k(e^k_i) = (0, 0, \dots)$$

adding these component wise gets us the equation

$$(c_1, c_2, \dots, c_k, 0, 0, \dots) = (0, 0, \dots).$$

Therefore $c_1 = \cdots = c_k = 0$, and the $(e^1_i), \dots, (e^k_n)$ are independent. Therefore we have contradiction, and C is not finite dimensional.

Perhaps a faster way to say this is that C has arbitrarily large sets of independent vectors in it, so there can be no finite basis.