Exam 2 will roughly cover 3.1-3.4, 3.6, and 4.1-4.4, in Olver and Shakiban.

Topics

- inner product on a real vector space (3.1)
- dot product on $\mathbb{R}^n$ (3.1)
- weighted dot product $\mathbb{R}^n$ (3.1)
- inner product on function vector spaces (3.1)
- norm from an inner product (3.1)
- Cauchy-Schwartz inequality (3.2)
- Triangle inequality (3.2)
- orthogonal vectors (3.2, 4.1)
- angle between two vectors (3.2)
- norm in general (3.1, 3.3)
- L^1, L^2, L^∞ norms on $\mathbb{R}^n$ and $C^0[a, b]$ (3.3)
- unit vectors (3.3)
- positive definite matrix (3.4)
- quadratic form $x^T K x$ (3.4)
- Gram matrix (3.4)
- complex number (3.6)
- complex conjugate (3.6)
- complex row reduction (3.6)
- orthogonal and orthonormal bases (4.1)
- Gram-Schmidt (4.2)
- alternate Gram-Schmidt (4.2)
- orthogonal matrix (4.3)
- QR factorization (4.3)
- vector orthogonal to a subspace (4.4)
- orthogonal projection (4.4)
- orthogonal subspaces (4.4)
- orthogonal complement $W^\perp$ (4.4)
- cokernel, coimage of a matrix (2.5)

Theorems

- Cauchy-Schwarz Inequality, Thm 3.5
- Triangle Inequality, Thm 3.9
- Theorem 3.27
- Proposition 3.31
- Theorem 3.34
- Lemma 4.2
- Proposition 4.4, Theorem 4.5
- Theorem 4.7, Theorem 4.9
- Proposition 4.19, Lemma 4.22, Proposition 4.23
- Theorem 4.32
- Proposition 4.40, Proposition 4.44
- Theorem 4.45

1. Let A be any square matrix in $\mathcal{M}_{n\times n}(\mathbb{R})$.

(a) (5 points) Show that the quadratic forms x^TAx and x^TA^Tx are equal.

(b) (5 points) Show that $K = \frac{1}{2}(A + A^T)$ is a symmetric matrix.

(c) (5 points) Conclude that it suffices to only consider quadratic forms of symmetric matrices by showing that $x^TAx = x^TKx$.

(d) (5 points) Prove that if K is positive definite, then every diagonal entry of A is positive.

Solution. (a) This follows from the fact that the dot product is symmetric.

$$x^TAx = x \cdot Ax = Ax \cdot x = (Ax)^Tx = x^TA^Tx$$

(b) This is symmetric by the fact that constants come out of transposes and that the transpose twice is the original matrix.

$$K^T = \frac{1}{2}(A + A^T)^T = \frac{1}{2}(A^T + (A^T)^T) = \frac{1}{2}(A + A^T) = K$$

(c) By the first two parts

$$x^TKx = x^T\left(\frac{1}{2}(A + A^T)\right)x = \frac{1}{2}\left(x^TAx + x^TA^Tx\right) = \frac{1}{2}\left(2x^TAx\right) = x^TAx.$$

(d) If K is positive definite, then $x^TKx > 0$ for all nonzero x. In particular, let $x = e_i$. On the one hand $e_i^TAe_i = a_{ii}$, the ith diagonal entry. But on then other hand $e_i^TAe_i = e_i^TKe_i > 0$. Therefore $a_{ii} > 0$.

2. Find the QR factorization of the matrix

$$\begin{pmatrix} -1 & -2 & 1 \\ 0 & 2 & 3 \\ 1 & 0 & 1 \end{pmatrix}.$$

Solution. Using the alternative G-S algorithm, we get

$$\begin{pmatrix} -1 & -2 & 1 \\ 0 & 2 & 3 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & \sqrt{2} & 0 \\ 0 & \sqrt{6} & 2\sqrt{\frac{2}{3}} \\ 0 & 0 & \frac{5}{\sqrt{3}} \end{pmatrix}.$$

3. Let $f(x) = 1$ and $g(x) = ax$ for $a \neq 0$ in the vector space $C^0[0,1]$ with inner product

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx.$$

(a) (10 points) Find the $a \in \mathbb{R}$ such that the angle between f and g is $\pi/6$, or 30 degrees.

(b) (10 points) Does your answer change if $f(x) = b$ for some other $b \neq 0, 1$? Explain why or why not.

Solution. (a) Recall that $\|1\|\,\|ax\|\cos(\theta) = \langle 1, ax\rangle$, so that

$$\cos(\theta) = \frac{\langle 1, ax\rangle}{\|1\|\,\|ax\|} = \frac{\int_0^1 ax\,dx}{\sqrt{\int_0^1 1\,dx}\sqrt{\int_0^1 a^2x^2\,dx}} = \frac{\frac{1}{2}a}{1\sqrt{\frac{a^2}{3}}} = \frac{\sqrt{3}}{2}\frac{a}{|a|}.$$

Note that if $a > 0$, then $a/|a| = 1$ and if $a < 0$ then $a/|a| = -1$. But $\cos\pi/6 = \sqrt{3}2$ so we want the positive solution. Therefore for all $a > 0$, then the angle between $f = 1$ and $g = ax$ is $\theta = \pi/6$.

(b) We can answer this question using the general principles of inner products. Indeed

$$\cos(\theta) = \frac{\langle b, ax\rangle}{\|b\|\,\|ax\|} = \frac{ab\langle 1, x\rangle}{|a||b|\,\|1\|\,\|x\|} = \frac{\sqrt{3}}{2}\frac{a}{|a|}\frac{b}{|b|}.$$

Therefore we see that a and b have the same sign, then the angle is 30 degrees, and if they have opposite sign, the angle is 150 degrees. This makes sense geometrically. The angle between v and w should be the same as with v and cw for $c > 0$. And if $c < 0$, then the angle becomes $\pi - \theta$. Scaling these vectors shouldn't change the angle unless the scale changes the sign. This principle is just being applied to functions in this problem.

4. Let v and w be independent vectors in $\mathbb{R}^n$. Let $v^\perp$ and $w^\perp$ denote the orthogonal subspaces of $\operatorname{span}(v)$ and $\operatorname{span}(w)$. Show that $\dim(v^\perp \cap w^\perp) = n - 2$.

Solution. Indeed $v^\perp \cap w^\perp = \operatorname{span}(v, w)^\perp$, namely the subspace of vectors orthogonal to both v and w. Putting v and w into the columns of an $n \times 2$ matrix A, we know that

$$v^\perp \cap w^\perp = \operatorname{span}(v, w)^\perp = \ker A^T.$$

Since v and w are independent, the rank of A^T is 2, and therefore the kernel has dimension $n - 2$ by rank nullity. This completes the proof.

5. Find an orthonormal basis for the subspace

$$W = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} \right\}.$$

Solution. This is the Gram-Schmidt process. We can do the normal version, and then divide the resulting orthogonal basis by the vectors' norms to get an orthonormal one. First

$$v_1 = w_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Then

$$v_2 = w_2 - \frac{w_2 \cdot v_1}{\|v_1\|^2} v_1 = \begin{pmatrix} 2 \\ -2 \\ 2 \\ 0 \end{pmatrix}.$$

Actually by inspection, the third vector is already orthogonal to these two. Therefore $v_3 = (0, 0, 0, -1)$. Therefore an orthonormal basis is

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

6. Let $P = I - uu^T$ where u is a unit vector in $\mathbb{R}^n$ with the dot product.

(a) (10 points) Show by direct computation that $P^2 = P$.

(b) (10 points) Compute $(\mathrm{img}(P))^\perp$.

Solution. (a) Note that since u is a unit vector, then $u^T u = 1$. Therefore

$$\begin{aligned} P^2 = (I - uu^T)(I - uu^T) = I - 2uu^T + (uu^T)(uu^T) &= I - 2uu^T + u(u^T u)u^T \\ &= I - 2uu^T + uu^T = I - uu^T = P. \end{aligned}$$

(b) By the relationship between the fundamental subspace, $(\mathrm{img}(P))^\perp = \mathrm{coker} P = \ker P^T$. First,

$$P^T = (I - uu^T)^T = I^T - (uu^T)^T = I - (u^T)^T u^T = I - uu^T = P.$$

Turns out P is symmetric. Therefore, we need to find $\ker P$. In fact $\mathrm{span}(u) = \ker P$. If we let $w \in \ker P$, then $(I - uu^T)w = 0$. But expanding this out gives that equation $w - uu^T w = 0$, which tells us that

$$W = u(u^T w) = au$$

where $a = u \cdot w$. Therefore w must be a multiple of u. Furthermore $u \in \ker P$, since

$$(I - uu^T)u = u - u(u^T u) = u - u = 0.$$

Therefore $\mathrm{img}(P)^\perp = \ker P$ is the span of u.

7. Find the distance between $v = (0, 3, -2, -2)$ and $w = (4, -1, 2, 1)$ in the following norms on $\mathbb{R}^4$.

(a) The L^2 norm

(b) The L^1 norm

(c) The L^∞ norm

Solution. This distance between two vectors is always $\|v - w\|$. To make this easier, $v - w = (-4, 4, -4, -3)$.

(a) $\|(-4, 4, -4, -3)\|_2 = \sqrt{16 + 16 + 16 + 9} = \sqrt{57}$
(b) $\|(-4, 4, -4, -3)\|_1 = |-4| + |4| + |-4| + |-3| = 15$
(c) $\|(-4, 4, -4, -3)\|_\infty = \max\{|-4|, |4|, |-4|, |3|\} = 4$

8. For the following statements, list whether they are true or false. If false, provide a counterexample.

(a) All matrices with positive entries are positive definite.

(b) Every orthogonal matrix Q has $\det Q = 1$.

(c) Given a vector $v \in V$ and a finite dimensional subspace $W \subseteq V$, then $\text{proj}_W(v) \perp v - \text{proj}_W(v)$.

(d) The set $\{e_1, e_2\}$ always forms an orthonormal basis of $\mathbb{R}^2$.

Solution. (a) False, $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ is not positive definite. The quadratic form is $x^2 + 4xy + y^2$, which is not positive when $(x, y) = (1, -1)$, since $q(1, -1) = 1^2 + -4 + 1^2 = -2$.
(b) False, the theorem states that $\det Q = \pm 1$. As a counterexample, $\det \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -1$.
(c) True! This is the definition.
(d) False, it depends on what inner product you use! For example, if $\langle \vec{x}, \vec{y} \rangle = 2x_1y_1 + 3x_2y_2$, then $\{e_1, e_\}$ is only an orthogonal basis, not an orthonormal one. Neither are unit vectors anymore. With this inner product $\|e_1\| = \sqrt{2}$ and $\|e_2\| = \sqrt{3}$.

9. (a) Find an orthonormal basis for the orthogonal complement of the kernel of the matrix

$$\begin{pmatrix} 1 & 0 & -1 & 1 & -1 \\ 1 & 2 & 0 & 0 & -1 \end{pmatrix}.$$

(b) Project the vector $v = (1, 1, 0, 0, 3)$ onto this subspace.

Solution. (a) What you should NOT do is row reduce A, find the kernel, and then calculate the orthogonal complement and then do G-S. This is way too much work. We know that for any matrix $\ker A^\perp = \text{coimg}\, A$. So all we need to do is find an orthonormal basis for the span of the rows. We already know a basis for the coimage, namely the rows of A. So this problem is asking you to do Gram-Schmidt to the vectors $w_1 = (1, 0, -1, 1, -1)$ and $w_2 = (1, 2, 0, 0, -1)$. This will yield

$$u_1 = \left(\tfrac{1}{2}, 0, -\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}\right) \quad u_2 = \left(\tfrac{1}{2\sqrt{5}}, \tfrac{2}{\sqrt{5}}, \tfrac{1}{2\sqrt{5}}, -\tfrac{1}{2\sqrt{5}}, -\tfrac{1}{2\sqrt{5}}\right).$$

(b) To project, we can use the projection formula for an orthonormal basis, which says that $\text{proj}_W v = \langle v, u_1 \rangle u_1 + \langle v, u_2 \rangle u_2$. Computing this using part (a), the coefficients are $c_1 = \langle v, u_1 \rangle = -1$ and $c_2 = \langle v, u_2 \rangle = \frac{1}{\sqrt{5}}$ to get

$$\text{proj}_W(v) = -u_1 + \frac{1}{\sqrt{5}} u_2 = \frac{1}{5} \begin{pmatrix} -2 \\ 2 \\ 3 \\ -3 \\ 2 \end{pmatrix}.$$

10. Prove that a matrix K is positive definite iff for all nonzero $v \in \mathbb{R}^n$ that the angle θ between v and Kv is acute, i.e. $|\theta| < \pi/2$.

Solution. Remember that an angle θ is acute iff $\cos(\theta) > 0$, since cos is positive only when $-\pi/2 < \theta < \pi/2$. Furthermore if θ is the angle between v and Kv, then θ is acute iff

$$\cos(\theta) = \frac{\langle v, Kv \rangle}{\|v\| \, \|Kv\|} = \frac{v \cdot Kv}{\|v\| \, \|Kv\|} = \frac{v^T Kv}{\|v\| \, \|Kv\|} > 0$$

So we need to show that K is positive definite iff

$$\frac{v^T K v}{\|v\| \, \|Kv\|} > 0$$

for all nonzero $v \in \mathbb{R}^n$.

This is a bit more straightforward. If K is positive definite, then $v^T K v > 0$. Furthermore $Kv \neq 0$, since being positive definite implies that K is invertible, which implies that $\ker K = 0$. Therefore this fraction is well defined (no dividing by 0), since $\|v\| > 0$ and $\|Kv\| > 0$. Since the numerator and denominator are both positive, then entire fraction is strictly positive as well.

Conversely, If the fraction is strictly positive, then it is well-defined (no dividing by 0). We know the denominator is then positive since it is the norms of some vectors, and norms are always positive. Finally since the whole fraction is positive, and so is the denominator, then the numerator must be positive for all $v \neq 0$. Therefore $v^T K v > 0$ for all $v \neq 0$ and by definition K is positive definite.