Topics

- $m \times n$ matrix
- ij^{th} entry in a matrix A, a_{ij}
- row vector vs. column vector
- zero matrix
- identity matrix
- diagonal matrix
- elementary matrix
- upper/lower triangular matrix
- pivots
- LU decomposition
- nonsingular matrix
- permutation matrix
- permuted LU decomposition
- matrix inverse
- matrix transpose
- Theorem 1.18
- Lemma 1.19 - Lemma 1.21
- Theorem 1.28
- Lemma 1.32
- symmetric matrix
- Reduced Row Echelon Form (RREF)
- matrix rank
- Interpreting solution from RREF, free variables
- Theorem 1.45
- vector spaces
- $\mathbb{R}^n$, polynomials, $C^0(\mathbb{R})$
- subspaces, defining properties of subspaces
- linear combination
- Span$(v_1, \ldots, v_n)$
- Spanning set of a vector space
- linearly independent vectors
- linearly dependent vectors
- Basis of a vector space, basis of a subspace
- Theoerem 2.21, Theorem 2.28, Theorem 2.29
- rank of a matrix
- Fundamental Subspaces: kernel, image, cokernel, coimage
- column space, row space, null space
- Proposition 2.41
- Theorem 2.49
- Fundamental Theorem of Linear Algebra (Equivalent Conditions of Invertibility)

1. Find all 2×2 matrices A which satisfy the equation

$$A^2 = 2I.$$

2. Compute (a) the inverses and (b) the (permuted if needed) LU decomposition of the matrices

$$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 3 \\ 1 & -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 5 & -1 & 2 \\ 3 & 2 & 1 \\ 0 & 2 & 0 \end{pmatrix}.$$

3. Let P^n be the vector space of polynomials of degree $\leq n$. Find the dimension of this vector space.

4. (a) Let $A = \begin{pmatrix} -1 & 3 \\ 1 & 2 \end{pmatrix}$ and $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$. Compute the expression $\vec{x}^T A \vec{x}$. (b) Consider the polynomial in two variables $2x^2 + xy + 3y^2$. Find a matrix B such that

$$2x^2 + xy + 3y^2 = \vec{x}^T B \vec{x}.$$

(c) Show that every polynomial of the form $ax^2 + bxy + cy^2$ can be written in the form $\vec{x}^T M \vec{x}$ where M is a symmetric matrix.

5. Let V be a vector space. Let U and W be subspaces of V. (a) Show that the intersection $U \cap W$ is a subspace. (b) Let $V = \mathbb{R}^4$. Find subspaces U and W such that $\dim U \cap W = 0$.

6. Define $M_{m \times n}(\mathbb{R})$ to be the set of $m \times n$ matrices with real entries. (a) Show that this is a vector space under the operations $A + B$, cA, where

$$(A + B)_{ij} = A_{ij} + B_{ij} \quad (cA)_{ij} = c(A)_{ij}.$$

What is the dimension of $M_{m \times n}(\mathbb{R})$?
(b) Recall that the trace of a square matrix A is the sum of its diagonal elements.

$$\operatorname{tr} A = \sum_i a_{i,i} = a_{1,1} + \cdots + a_{n,n}$$

Show that the set of matrices A with $\operatorname{tr} A = 0$ is a subspace of the vector space $M_{n \times n}(\mathbb{R})$. (Optional Challenge: What's the dimension of the subspace of trace 0 matrices?)

7. (a) Determine whether the vector $\begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}^T$ is in the span of the vectors

$$\begin{pmatrix} -1 \\ 1 \\ -2 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -1 \\ 2 \end{pmatrix} \begin{pmatrix} 7 \\ -3 \\ 0 \\ 2 \end{pmatrix}.$$

(b) What is the dimension of the span of these 3 vectors? Can the 3 vectors possibly form a basis of $\mathbb{R}^4$?

8. Consider the vector space $C^0(\mathbb{R})$ of continuous functions on $\mathbb{R}$. Show that the functions $f(x) = \cos(2x)$, $g(x) = \cos^2(x)$ and $h(x) = 1$ are linearly dependent in this vector space using a trig identity.

9. (a) Find the numbers a such that the columns of the following matrix form a basis of $\mathbb{R}^3$.

$$A = \begin{pmatrix} a & 1 & 2 \\ 0 & a & 1 \\ -1 & 2 & a \end{pmatrix}$$

(b) For what a is the rank $A = 1$? How about rank $A = 2$?

10. Show that two vectors v_1, v_2 in $\mathbb{R}^2$ form a basis when v_1 is not a multiple of v_2.

11. Suppose a matrix M has 5 columns, labeled $v_1, \ldots, v_5$. Suppose that M has the following RREF form.

$$\begin{pmatrix} 1 & 0 & 3 & -1 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) Find the rank of M. (b) Which columns of M form a basis of the image of M. (c) If possible, write v_3, v_4, and v_5 in terms of the vectors before it. (d) Find $\ker M$ and the nullity of M. (e) How many independent rows does M have? (f) Is M invertible?

12. Find the solution sets to the following linear systems where possible. (If no solution, say "no solution".)

a) $$\begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

b) $$\begin{aligned} x + y + z &= 0 \\ x - 2y + z &= 1 \\ -x + y + z &= 2 \end{aligned}$$

c) $$\begin{aligned} -x + y - z + w &= 0 \\ x - y - z + w &= 0 \\ y + 2z &= 1 \end{aligned}$$

13. (Challenge, harder than test problem) Define $\mathbb{R}^\infty$ to be the set of all infinite sequences of real numbers. (a) Show that the set of all convergent subsequences is a subspace. (b) Determine whether C is finite or infinite dimensional.