1. Find all 2×2 matrices A which satisfy the equation

$$A^2 = 2I.$$

Solution. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then the above equations becomes

$$\begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Then $a, d = \pm\sqrt{2 - bc}$ and either $b, c = 0$ or $a = -d$. If $b, c = 0$, then the matrix has the form

$$\begin{pmatrix} \pm\sqrt{2} & 0 \\ 0 & \pm\sqrt{2} \end{pmatrix}.$$

Else if $a = -d$, then $bc \leq 2$ to keep the matrix real, so that the matrix has the form

$$\begin{pmatrix} \pm\sqrt{2 - bc} & b \\ c & \mp\sqrt{2 - bc} \end{pmatrix}.$$

2. Compute the permuted LDV decomposition of the matrix

$$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 3 \\ 1 & -1 & 0 \end{pmatrix}.$$

Determine the rank and dimension of the kernel as well.

Solution. Note that there is more than one way to compute a permuted LU decomposition.
(a)

$$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 3 \\ 1 & -1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} -3 & 2 & 1 \\ -3 & 2 & 0 \\ 2 & -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 3 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 1/2 \end{pmatrix}$$

(b)

$$\begin{pmatrix} -1 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} -1/3 & 0 & 2/3 \\ 1/3 & 0 & 1/3 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(c)

$$\begin{pmatrix} 5 & -1 & 2 \\ 3 & 2 & 1 \\ 0 & 2 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 2 & -5/2 \\ 0 & 0 & 1/2 \\ 3 & -5 & 13/2 \end{pmatrix}$$

$$\begin{pmatrix} 5 & -1 & 2 \\ 3 & 2 & 1 \\ 0 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3/5 & 1 & 0 \\ 0 & 10/13 & 1 \end{pmatrix} \begin{pmatrix} 5 & -1 & 2 \\ 0 & 13/5 & -1/5 \\ 0 & 0 & 2/13 \end{pmatrix}$$

3. Let P^n be the vector space of polynomials of degree $\leq n$. Find the dimension of this vector space.

Solution. The dimension of P^n is $n+1$. We can show that the polynomials $\{1, x, \ldots, x^n\}$ form a basis of P^n. First we can argue that these monomials are independent. Suppose some linear combination of them were 0, namely

$$c_0 + c_1 x + \ldots c_n x^n = 0.$$

If one of the $c_i \neq 0$ then we would have an actual polynomial on our hands, which is definitely not the 0 polynomial by definition. So it must be that $c_i = 0$ for all i. (Kind of tautological I know, that's just how monomials work, they never cancel with each other.) These functions span P^n by definition; all degree n or less polynomials are a linear combination of $\{1, x, \ldots, x^n\}$. Thus they form a basis. To find the dimension of a vector space, we can just count the number of vectors in any basis. Therefore $\dim P^n = n+1$.

4. (a) Let $A = \begin{pmatrix} -1 & 3 \\ 1 & 2 \end{pmatrix}$ and $x = \begin{pmatrix} x \\ y \end{pmatrix}$. Compute the expression $x^T A x$. (b) Consider the polynomial in two variables $2x^2 + xy + 3y^2$. Write find a matrix B such that polynomial in the form

$$2x^2 + xy + 3y^2 = x^T B x.$$

(c) Show that every polynomial of the form $ax^2 + bxy + cy^2$ can be written in the form $x^T M x$ where M is a symmetric matrix.

Solution. (a) $-x^2 + 4xy + 2y^2$ (b) $B = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$ (c) The matrix M has the form

$$\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}.$$

5. Let V be a vector space. Let U and W be subspaces of V. (a) Show that the intersection $U \cap W$ is a subspace. (b) Let $V = \mathbb{R}^4$. Find subspaces U and W such that $\dim U \cap W = 0$.

Solution. (a) Let $v, w \in U \cap W$. In particular both v and w are in both U and W. First $U \cap W$ is nonempty since $0 \in U$ and $0 \in W$. Then $v + w \in U$ since U is a subspace and similarly for W. Thus $v + w \in U \cap W$. Finally, if $c \in \mathbb{R}$, then $cv \in U$ since U is a subspace and similarly for W. Thus $cv \in U \cap W$.

(b) If we let $U = \operatorname{span}\{(1,0,0,0)^T, (0,1,0,0)^T\}$ and $W = \operatorname{span}\{(0,0,1,0)^T, (0,0,0,1)^T\}$. These subspaces have trivial intersection since all of the vectors are linearly independent.

6. Define $M_{m \times n}(\mathbb{R})$ to be the set of $m \times n$ matrices with real entries. (a) Show that this is a vector space under the operations $A + B$, cA, where

$$(A+B)_{ij} = A_{ij} + B_{ij} \quad (cA)_{ij} = c(A)_{ij}.$$

What is the dimension of $M_{m \times n}(\mathbb{R})$?
(b) Recall that the trace of a square matrix A is the sum of its diagonal elements.

$$\operatorname{tr} A = \sum_i a_{i,i} = a_{1,1} + \cdots + a_{n,n}$$

Show that the set of matrices A with $\operatorname{tr} A = 0$ is a subspace of the vector space $M_{n\times n}(\mathbb{R})$. (Optional Challenge: What's the dimension of the subspace of trace 0 matrices?)

Solution. (a) Proving the vector space properties for $M_{m\times n}(\mathbb{R})$ is essentially the same as it is for $\mathbb{R}^n$. I'll highlight the important parts.

For the zero vector in $M_{m\times n}(\mathbb{R})$, that role is played by the matrix of all 0's, which I will denote also by 0. Indeed entrywise addition by 0 is trivial; $A + 0 = 0 + A = A$. Similarly, the negative vector $-A$ can be found by just making all the entries of A negative, i.e. $(-A)_{ij} = -(A)_{ij}$. Then it is clear that $A + (-A) = 0$. The rest of the properties follow as you would exepct. (Not quite as unobvious as 2.1.2.)

As for the dimension of $M_{m\times n}(\mathbb{R})$, we need to find a basis of this vector space. The intuition here is that it's basically the same as $\mathbb{R}^{mn}$ except instead writing a giant mn-sized column vector, we write it in a grid instead. So we claim that $\dim M_{m\times n}(\mathbb{R}) = mn$. To show this we need to find a basis and count the number of elements in it.

We can find a basis by making "standard basis matrices" just like we made standard basis vectors. Remember e_i was a vector of all 0's, except a 1 in the ith spot. Now we can do that with matrices. Define a matrix $E^{(ij)}$ by letting $E^{(ij)}_{ij} = 1$ and 0 otherwise. Here are the standard basis matrices for the 2 by 2 case for example.

$$E^{(11)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad E^{(12)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad E^{(21)} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad E^{(22)} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

The set $\{E^{ij}\}$ forms a basis of $M_{m\times n}(\mathbb{R})$. They are linearly independent since they all have 1's in different spots, and they span since given any matrix A, we can write

$$A = \sum_{ij} (A_{ij}) E^{(ij)}.$$

For example

$$\begin{pmatrix} -2 & 3 \\ 1 & 5 \end{pmatrix} = -2E^{(11)} + 3E^{(12)} + 1E^{(21)} + 5E^{(22)}.$$

Since the standard basis matrices are independent and span, they form a basis. If we count them up, there are mn basis vectors so $\dim M_{m\times n}(\mathbb{R}) = mn$ as desired.

(b) First the zero matrix 0 has no trace, so it satisfies the first property. Let A and B be matrices with $\operatorname{tr}(A) = 0 = \operatorname{tr}(B)$. For additive closure, note that $\operatorname{tr}(A+B) = \operatorname{tr}(A) + \operatorname{tr}(B)$, since matrices add component wise. Therefore $\operatorname{tr}(A + B) = 0 + 0 = 0$. To show that scalar multiplication is closed, note that $\operatorname{tr}(cA) = \sum_i c(A)_{i,i} = c\operatorname{tr}(A) = 0$. Thus the trace zero matrices form a subspace.

Optional Challenge solution: As for the dimension, we can be really really clever and apply rank-nullity. Pretend that $M_{n\times n}(\mathbb{R})$ is just a copy of $\mathbb{R}^{n^2}$ by writing all the entries of the matrix in one big column; maybe like first row on top of the second row, etc. Like this

$$A = (a_{11}, a_{12}, \ldots, a_{1n}, a_{21}, \ldots, a_{nn})^T.$$

Now you can think of the trace as a matrix multiplication operation as follows Since $\operatorname{tr}(A) = a_{11} + \cdots + a_{nn}$ this kind of looks like a matrix T times the giant n^2-vector above. If you're a little clever, you can see "trace matrix" T will have 1's for every a_{ii} entry and 0's otherwise. To be explicit, the 3×3 case looks like this.

$$T(A) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 1 \\ a_{12} \\ a_{13} \\ a_{21} \\ a_{22} \\ a_{23} \\ a_{31} \\ a_{32} \\ a_{33} \end{pmatrix} = a_{11} + a_{22} + a_{33} = \operatorname{tr} A.$$

Taking the trace is the same as multiplying by that row matrix, which we can think of as the matrix representing the trace operation. We'll call this $1 \times n^2$ matrix T. So the set of trace 0 matrices, is just the set of matrices such that $\operatorname{tr}(A) = TA = 0$ can be considered as $\ker T$! So all we have to do is compute the dimension of the kernel of this $1 \times n^2$ matrix.

Well, it's a $1 \times n^2$ matrix, so the rank can be at most 1. In fact the rank is 1, since we have a nonzero column. (It's already in RREF actually!). Since $\operatorname{rank}(T) = 1$, then

$$\dim \ker T = \text{number of columns} - \operatorname{rank} T = n^2 - 1.$$

Therefore, the set of trace 0 matrices is $n^2 - 1$ dimensional.

7. (a) Determine whether the vector $\begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}^T$ is in the span of the vectors

$$\begin{pmatrix} -1 \\ 1 \\ -2 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -1 \\ 2 \end{pmatrix} \begin{pmatrix} 7 \\ -3 \\ 0 \\ 2 \end{pmatrix}.$$

(b) What is the dimension of the span of these 3 vectors? Can the 3 vectors possibly form a basis of $\mathbb{R}^4$?

Solution. Put the three vectors in the first three columns of a matrix, and make $(1, 2, 3, 4)^T$ the last column of the matrix. If the system has a free column in the fourth column, then the last vector is dependent on the other three. If it doesn't, then the fourth vector is independent of the other three.

This matrix actually row reduces to the identity.

$$\begin{pmatrix} -1 & 0 & 7 & 1 \\ 1 & 0 & -3 & 2 \\ -2 & -1 & 0 & 3 \\ 0 & 2 & 2 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Therefore the vector $(1, 2, 3, 4)^T$ is not in the span of the other 3. Furthermore, the first 3 vectors can't possibly be a basis of $\mathbb{R}^4$. First of all, all bases of $\mathbb{R}^4$ have 4 vectors in them. But additionally, we just showed that these 3 vectors don't span, so they can't be a basis.

8. Consider the vector space $C^0(\mathbb{R})$ of continuous functions on $\mathbb{R}$. Show that the functions $f(x) = \cos(2x)$, $g(x) = \cos^2(x)$ and $h(x) = 1$ are linearly dependent in this vector space.

Solution. The double angle formula is

$$\cos(2x) = 2\cos^2(x) - 1.$$

Rewritten, this is a linear relationship between the functions so that $f(x)-2g(x)+h(x)=0$ and they are dependent.

9. (a) Find the numbers a such that the columns of the following matrix form a basis of $\mathbb{R}^3$.

$$A=\begin{pmatrix} a & 1 & 2\\ 0 & a & 1\\ -1 & 2 & a\end{pmatrix}$$

(b) For what a is the rank $A=1$? How about rank $A=2$?

Solution. (a) By the main theorem for square matrices, $\det A=0$ iff the columns of A do not form a basis. Taking the determinant, we obtain that $\det A=a^3-1=0$. The only solution in the reals is $a=1$. So when $a\neq 1$, the columns form a basis of $\mathbb{R}^3$.

(b) We know that rank $A=3$ when $a\neq 1$. So we just have to check what the rank is when $a=1$. In this case, the matrix row reduces to

$$\begin{pmatrix} 1 & 0 & 1\\ 0 & 1 & 1\\ 0 & 0 & 0\end{pmatrix}$$

which is rank 2 since it has 2 pivots. Alternatively, you could have seen that the first two columns of A are independent one is not a multiple of the other. Either way, the rank is 1 for no $a\in\mathbb{R}$, and for $a=1$, the rank is 2.

10. Show that two vectors v_1, v_2 in $\mathbb{R}^2$ form a basis when v_1 is not a multiple of v_2.

Solution. Suppose for contradiction that $\{v_1.v_2\}$ form a basis and $v_2=cv_1$. But this can be rewritten as $cv_1+(-1)v_2=0$ so that v_1 and v_2 have a linear relationship, and therefore are not independent. Then they can't be a basis. This is a contradiction, so $v_2\neq cv_1$ for any c. They are not multiples of each other.

11. Suppose a matrix M has 5 columns, labeled $v_1,\ldots,v_5$. Suppose that M has the following RREF form.

$$\begin{pmatrix} 1 & 0 & 3 & -1 & 0\\ 0 & 1 & -2 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}$$

(a) Find the rank of M. (b) Which columns of M form a basis of the image of M. (c) If possible, write v_3, v_4, and v_5 in terms of the vectors before it. (d) Find $\ker M$ and the nullity of M. (e) How many independent rows does M have? (f) Is M invertible?

Solution. (a) $\text{rank}(A)=$ number of leading 1's $=3$. (b) Since the 1st, 2nd, and 5th columns have leading 1's in the RREF, then v_1, v_2, v_5 form a basis of the image of M. (c) From the RREF $v_3=3v_1+(-2)v_2$ and $v_4=(-1)v_1+1v_2$. The vector v_5 is independent, so it is not possible. (d) From the RREF, we know that there are 2 free columns. If we label the columns by the variables, x, y, z, w, u, then z, w are free. The rows of the the RREF tell us the equations between the variables, namely $x=-3z+w$, $y=2z-w$ and $u=0$. Therefore vectors in $\ker M$ have the form

$$\begin{pmatrix} x \\ y \\ z \\ w \\ u \end{pmatrix} = \begin{pmatrix} -3z+w \\ 2z-w \\ z \\ w \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} z + \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} w$$

and $\ker M = \text{span}\{(-3,2,1,0,0),(1,-1,0,1,0)\}$. The nullity is the dimension of the kernel, so in this case 2. Alternatively, we could used rank-nullity since we already knew the rank. (e) Since M has 3 independent columns, it also has 3 independent rows. Remember $\text{rank}(A) = \text{rank}(A^T)$. (f) M is not invertible, since $\text{rank}(M) \neq 5$.

12. Find the solution sets to the following linear systems where possible. (If no solution, say "no solution".)

a) $$\begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

b) $$\begin{aligned} x+y+z &= 0 \\ x-2y+z &= 1 \\ -x+y+z &= 2 \end{aligned}$$

c) $$\begin{aligned} -x+y-z+w &= 0 \\ x-y-z+w &= 0 \\ y+2z &= 1 \end{aligned}$$

Solution. (a) The matrix $\begin{pmatrix} 1 & -1 & -1 \\ -2 & 2 & 1 \end{pmatrix}$ has RREF form $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. So this system is inconsistent and has no solution. You can think of the constant vector $(-1,1)$ as lying outside of the image of $\begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}$.

(b) $(x,y,z) = (-1,-1/3,-4/3)$

(c) $$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ 1 \\ 1 \end{pmatrix} w + \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

13. Define $\mathbb{R}^\infty$ to be the set of all infinite sequences of real numbers. (a) Show that the set of all convergent subsequences is a subspace. (b) Determine whether C is finite or infinite dimensional.

Solution. First of all, the way I phrased the question doesn't tell you how $\mathbb{R}^\infty$ is a vector space. We can write a sequence as a tuple that just never ends.

$$(a_1, a_2, \ldots)$$

You can add these like vectors in $\mathbb{R}^n$. They add component-wise, and scalar multiply component-wise as well. These operations satisfy the 7 axioms.

(a) Let $(a_i) = (a_1, a_2, \dots)$ and $(b_i) = (b_1, b_2, \dots)$ be convergent sequences. Since they're convergent, let $(a_i) \to a$ and $(b_i) \to b$. Recall that the sum of two convergent sequences is also convergent, so that $(a_i + b_i) \to a + b$. Thus C is closed under addition.

Given a scalar c, it is clear that $c(a_i) = (ca_i) \to ca$. Therefore C is closed under scalar multiplication as well. The set C is also nonempty (since $(0, 0, \dots) \in C$), and therefore C is a subspace of $\mathbb{R}^\infty$.

(b) This subspace is infinite dimensional. Assume for contradiction that there exists a finite basis $\{(x_i)_1, (x_i)_2, \dots, (x_i)_n\}$, so that $\dim C = n$. Then by Theorem 2.31, any set of sequences $\{(y_i)_1, \dots, (y_i)_k\}$ is linearly dependent when $k > n$. We can show that this leads to a contradiction by finding k linearly independent convergent sequences for $k > n$.

Pick any number $k > n$. Let (e^i_j) be the sequence defined by

$$\begin{cases} e^i_j = 0 & i \neq j \\ e^i_i = 1 \end{cases}$$

Here i is not an exponent. It is in index, I'm just putting where the exponent usually goes because there was already another index in the subscript. For example

$$(e^1_j) = (1, 0, 0, \dots) \quad \text{and} \quad (e^3_j) = (0, 0, 1, 0, 0, \dots).$$

These are essentially the standard basis vectors, but now they are sequences instead.

First, note that $(e^j_i) \to 0$ for all j. This is true since if we let $N > j$, then for all $n > N$, $|e^j_n - 0| = 0 < \epsilon$ for all $\epsilon > 0$. (Oops I used n twice, different n here.) So all of our "standard basis sequences" are convergent to zero, and $(e^j_i) \in C$.

Now consider the set of sequences

$$\{(e^1_i), (e^2_i), \dots, (e^k_i)\}$$

where $k > \dim C = n$ as you recall. By Lemma 2.31, this set of vectors should be dependent, since $k > \dim C$. But we can show that they are independent. For given a linear combination

$$c_1(e^1_i) + \cdots + c_k(e^k_i) = (0, 0, \dots)$$

adding these component wise gets us the equation

$$(c_1, c_2, \dots, c_k, 0, 0, \dots) = (0, 0, \dots).$$

Therefore $c_1 = \cdots = c_k = 0$, and the $(e^1_i), \dots, (e^k_n)$ are independent. Therefore we have contradiction, and C is not finite dimensional.

Perhaps a faster way to say this is that C has arbitrarily large sets of independent vectors in it, so there can be no finite basis.