Textbook: 1.1.1c, 1.2.1, 1.2.4c, 1.2.5ab, 1.2.7abcd, 1.3.1c, 1.3.3ab, 1.3.16ab, 1.3.21ad, 1.5.3ad, 1.5.19, 1.5.27, 1.6.3, 1.6.19

Extra Problem #1: Consider the matrix
$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$
Find all real 2×2 matrices that commute with A.

Hint for 1.6.3: Remember that the basic formula is $(AB)^T = B^T A^T$ and not $A^T B^T$. So this problem is having you figure out what needs to happen for $(AB)^T = A^T B^T$ to be true. (And it turns out $AB = BA$ needs to be true.) Perhaps ask yourself when you are solving this problem, if $AB = BA$, does $B^T A^T = A^T B^T$?

Solution (1.2.1). (a) 3×4, (b) 7, (c) 6, (d) $\begin{pmatrix} -2 & 0 & 1 & 3 \end{pmatrix}$, (e) $\begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix}$

Solution (Extra Problem). Let $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If B commutes with A, then $AB = BA$. Expanding this equation, we obtain
$$\begin{pmatrix} a & a+b \\ c & c+d \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix}.$$
By equating the corresponding entries, we see that $a = d$ and $c = 0$. So therefore if B commutes with this particular matrix A, then it is of the form
$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}.$$
Conversely it is easy to check that all matrices of this form commute with A.

Solution (1.3.21c). The matrix $\begin{pmatrix} -1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix}$ can be row reduced to the upper triangular matrix $U = \begin{pmatrix} -1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ by the row operations $r_1' = r_1 + r_2$ and $r_3' = -r_1 + r_3$ in that order. The matrix L keeps track of the inverses of these row operations, so $(L)_{21} = -1$ from the first operation and $(L)_{31} = 1$ from the second. Thus,
$$\begin{pmatrix} -1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Solution (1.5.19). First, $A \sim A$ by $A = I^{-1} A I$ where I is the identity matrix. We can find any matrix that makes them similar, and the identity fits.

Second, assume $A \sim B$, so that $B = S^{-1} A S$. Then rearranging this equation yields
$$A = SBS^{-1} = (S^{-1})^{-1} B S^{-1}.$$
Therefore $B \sim A$ by the matrix S^{-1}.

Finally assume $A \sim B$ and $B \sim C$. Let $B = S^{-1} A S$ and $C = T^{-1} B T$. Substituting the first equation into the second gives us
$$C = T^{-1} S^{-1} A S T = (ST)^{-1} A (ST).$$
Therefore $A \sim C$ by the matrix ST.

Solution (1.6.3). Assume A, B are square commuting. Then their transposes also commute, for $AB = BA$ implies $(AB)^T = (BA)^T$, which is $B^T A^T = A^T B^T$. Therefore

$$(AB)^T = B^T A^T = A^T B^T.$$

Conversely, assume $(AB)^T = A^T B^T$. First, we show that A and B are square. Let A be $m \times n$, and B be $n \times p$. The n dimensions agree since AB exists as a matrix product. But since $A^T B^T$ exists as a product, then $m = p$. Furthermore Then $(AB)^T$ has dimensions $p \times m$ and $A^T B^T$ has dimension $n \times n$. Since these matrices are equal, we can conclude that $m = n = p$, so these matrices are square.

Now we show they commute. Indeed

$$AB = ((AB)^T)^T = (A^T B^T)^T = (B^T)^T (A^T)^T = BA.$$

Solution (1.6.19). Let A be a symmetric matrix. Then we can show that indeed A^2 is also symmetric. Remember by definition symmetric matrix means $A^T = A$. So we can show that $(A^2)^T = A^2$ to finish the problem. Indeed since $A^T = A$, then

$$(A^2)^T = A^T A^T = AA = A^2.$$

You can also do this entry by entry, note that

$$(A^2)_{ij} = \sum_{k=1}^{n} A_{ik} A_{kj} = \sum_{k=1}^{n} A_{ki} A_{jk} = \sum_{k=1}^{n} A_{jk} A_{ki} = (A^2)_{ji}.$$

Solution (1.3.1c). We form the augmented matrix out of the system and row reduce.

$$\left(\begin{array}{ccc|c} 2 & 1 & 2 & 3 \\ -1 & 3 & 3 & -2 \\ 4 & -3 & 0 & 7 \end{array}\right)$$

$$r_2' = r_1 + 2r_2 \qquad \left(\begin{array}{ccc|c} 2 & 1 & 2 & 3 \\ 0 & 7 & 8 & -1 \\ 4 & -3 & 0 & 7 \end{array}\right)$$

$$r_3' = \tfrac{-1}{5}(-2r_1 + r_3) \qquad \left(\begin{array}{ccc|c} 2 & 1 & 2 & 3 \\ 0 & 7 & 8 & -1 \\ 0 & 1 & 4/5 & -1/5 \end{array}\right)$$

$$r_3' = r_2 - 7r_3 \qquad \left(\begin{array}{ccc|c} 2 & 1 & 2 & 3 \\ 0 & 7 & 8 & -1 \\ 0 & 0 & 12/5 & 2/5 \end{array}\right)$$

$$r_3' = \tfrac{5}{12}r_3 \qquad \left(\begin{array}{ccc|c} 2 & 1 & 2 & 3 \\ 0 & 7 & 8 & -1 \\ 0 & 0 & 1 & 1/6 \end{array}\right)$$

$$r_2' = \tfrac{1}{7}r_2 \qquad \left(\begin{array}{ccc|c} 2 & 1 & 2 & 3 \\ 0 & 1 & 8/7 & -1/7 \\ 0 & 0 & 1 & 1/6 \end{array}\right)$$

$$r_2' = \tfrac{-8}{7}r_3 + r_2 \qquad \left(\begin{array}{ccc|c} 2 & 1 & 2 & 3 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & 1/6 \end{array}\right)$$

$$r_1' = -2r_3 + r_1 \qquad \left(\begin{array}{ccc|c} 2 & 1 & 0 & 8/3 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & 1/6 \end{array}\right)$$

$$r_1' = -r_2 + r_1 \qquad \left(\begin{array}{ccc|c} 2 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & 1/6 \end{array}\right)$$

$$r_1' = \tfrac{1}{2}r_1 \qquad \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3/2 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & 1/6 \end{array}\right)$$

Thus $u = 3/2$, $v = -1/3$, and $w = 1/6$.