Textbook: 8.2.19ab, 8.2.20, 8.3.2aceg, 8.3.1af, 8.5.1ad, 8.5.2ac

Solution (8.2.19ab). Suppose that λ is an eigenvalue of A. (a) Then $c\lambda$ is an eigenvalue of cA since

$$(cA)v = c(Av) = c(\lambda v) = (c\lambda)v$$

where v is an eigenvector. (b) Similarly $\lambda + d$ is an eigenvalue of $A + dI$ since

$$(A + dI)v = Av + dIv = \lambda v + dv = (\lambda + d)v.$$

Solution (8.2.20). Suppose λ is an eigenvalue of A. Suppose it has eigenvector v. Then λ^2 is an eigenvalue of A^2 with the same eigenvector, since

$$A^2 v = A(Av) = A(\lambda v) = \lambda(Av) = \lambda^2 v.$$

Solution (8.3.2e). I'll do (e) as an example. The eigenvalues for the matrix $\begin{pmatrix} 4 & -1 & -1 \\ 0 & 3 & 0 \\ 1 & -1 & 2 \end{pmatrix}$ are $\lambda = 3, 3, 3$. So the algebraic multiplicity of $\lambda = 3$ is 3 since it repeats three times. The corresponding eigenspace is

$$V_3 = \ker(A - 3I) = \ker \begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 1 & -1 & -1 \end{pmatrix}.$$

Our usual RREF algorithm for finding the kernel of a matrix tells us that it has a basis

$$V_3 = \operatorname{span} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

We have 2 basis vectors so the $\dim V_3 = 2$ and therefore the geometric multiplicity is 2. The eigenvector $\lambda = 3$ is not complete since the algebraic multiplicity is 3, but the geometric multiplicity is 2. We would require one generalized eigenvector.

Solution (8.5.1a). The characteristic polynomial of

$$\begin{pmatrix} 2 & 6 \\ 6 & -7 \end{pmatrix}$$

is

$$\lambda^2 + 5\lambda - 50 = 0.$$

This factors in $(\lambda - 5)(\lambda + 10) = 0$ so the eigenvalues of $\lambda = 5$ and $\lambda = -10$. These are distinct eigenvalues of a symmetric matrix, so whatever eigenvectors we find will be orthogonal naturally! Indeed

$$V_5 = \ker(A - 5I) = \operatorname{span} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

and

$$V_{-10} = \ker(A + 10I) = \operatorname{span} \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

As the theorem says, these eigenspaces are orthogonal, so to make an orthonormal eigenvector basis of $\mathbb{R}^2$ we just need to normalize.

$$u_1 = \tfrac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad u_2 = \tfrac{1}{\sqrt{5}} \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

Solution (8.5.2a). Recall that a symmetric matrix is positive definite if and only if its eigenvalues are all strictly positive. So if they're all positive, the matrix is positive definite. If one or more λ is 0 or less, then it's not positive definite. In particular, consider the matrix

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 3 \end{pmatrix}.$$

Computing the eigenvalues using the usual $\det A - \lambda I = 0$, we get that $\lambda = \frac{1}{2}(5 \pm \sqrt{17})$. We know that $\frac{1}{2}(5 + \sqrt{17}) > 0$ since it is the sum of positive numbers. But $\lambda = \frac{1}{2}(5 - \sqrt{17})$ is also positive since $16 < 17 < 25$ so $4 < \sqrt{17} < 5$. In any event $5 - \sqrt{17} > 0$ so all eigenvalues are positive, so $K = \begin{pmatrix} 2 & -2 \\ -2 & 3 \end{pmatrix}$ is positive definite. To connect back to chapter 3, we know that

$$\langle v, w \rangle = v^T \begin{pmatrix} 2 & -2 \\ -2 & 3 \end{pmatrix} w = 2v_1w_1 - 2v_1w_2 - 2v_2w_1 + 3v_2w_2$$

defines an inner product since K is positive definite!