Textbook: 1.4.21ace, 1.8.1be, 1.8.4, 1.8.23c, 1.8.24, 2.1.1, 2.1.12, 2.2.1, 2.2.2abc

Solution (1.8.1e). The augmented matrix for the system is

$$\left(\begin{array}{cccc|c} 1 & -2 & 2 & -1 & 3 \\ 3 & 1 & 6 & 11 & 16 \\ 2 & -1 & 4 & 1 & 9 \end{array}\right)$$

and row reduction yields

$$\left(\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 5 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right).$$

The number of leading 1s is less than the number of columns, so that there is on free variable and an infinite number of solutions. The 3rd column has no leading 1, or no pivot, so z is a free variable. We know from the reduced matrix that $y = 1$, $w = 0$, and $z = s$ Furthermore, $x = -2z + 5$. The solution in vector form is

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

The solution set has the shape of a line, starting at $(5, 1, 0, 0)$ and going in the direction of $(-2, 0, 1, 0)$. In more advanced language, the kernel of the original matrix A is dimension 1 since the rank is 3, so the solution set is an affine subspace of dimension 1 as well.

Solution (1.8.4). Given the augmented matrix

$$\left(\begin{array}{ccc|c} a & 0 & b & 2 \\ a & 2 & a & b \\ b & 2 & a & a \end{array}\right)$$

we can row reduce to

$$\left(\begin{array}{ccc|c} a & 0 & b & 2 \\ 0 & 2 & a-b & b-2 \\ b & 0 & b & 2-b+a \end{array}\right)$$

using the steps $r_2' = -r_1 + r_2$ and $r_3' = -r_2 + r_3$. Assume that $a \neq 0$, so that we have at least 2 pivots, a and 2. This system has a unique solution solution iff the last pivot doesn't get cancelled. However this happens iff $a \neq b$ or $b \neq 0$. In one case, the a cancels the b in the (31) entry or if $b = 0$ the bottom row was all 0's in the first place. Now if $a = 0$, then the original system has matrix

$$\begin{pmatrix} 0 & 0 & b \\ 0 & 2 & 0 \\ b & 2 & 0 \end{pmatrix}.$$

By swapping the first and third rows, we can see that this has three pivots iff $b \neq 0$, so we have a nonunique solution when $a = b = 0$.

Combining both cases, we see that have a nonunique solution when $a = b$ or $b = 0$.

Now we can get more specific to see when we have infinite solutions or no solutions. If $b = 0$, then the bottom row is $\begin{pmatrix} 0 & 0 & 0 & 2+a \end{pmatrix}$. So the system has no solution if $a \neq -2$, and has an infinite number of solutions if $a = -2$.

If $a = b$, then the system reduces to

$$\left(\begin{array}{ccc|c} a & 0 & a & 2 \\ 0 & 2 & 0 & a-2 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

This system has an infinite number of solutions, since the z column is free.

Solution (1.8.24). From Theorem 1.47, a homogeneous system has trivial solution iff the matrix is nonsingular. However, an upper triangular matrix is nonsingular iff U has n pivots. But the only way an upper triangular matrix can have n pivots is if the diagonal entries are all nonzero. Therefore the system has trivial solution iff the diagonal entries are all nonzero.

Another proof is as follows. A matrix is nonsingular iff $\det U \neq 0$. But the determinant of an upper triangular matrix is the product of the diagonal entries (easy to check using permutation formula). Therefore the system will have nontrivial solution only a diagonal entry is 0 making the determinant 0.

Here is a more computational approach you can take to understand the situation. Assume that U has a nontrivial solution $v = (v_1 \dots v_n)^T \neq 0$. Then assume for contradiction that U has all nonzero entries on the diagonal, we know that $u_{nn}v_n = 0$. Since $u_{nn} \neq 0$, we must conclude that $v_n = 0$. Similarly, now we know that the previous row now is $u_{n-1,n-1}v_{n-1} = 0$. Again we conclude that $v_{n-1} = 0$. So by induction $v = 0$. But this is a contradiction, since $v \neq 0$. Thus one of the $u_{ii} = 0$. In other words, the back substitution algorithm tells us that the solution must be trivial of $u_i \neq 0$.

The converse is annoying with this method, so that's why the pivot and determinant proofs are easiest. We'll understand this result another way when we cover eigenvalues.

Solution (2.1.1). We verify that $\mathbb{C}$ is a vector space over $\mathbb{R}$. Most of these follows just from the definition of complex addition and multiplication. The addition is commutative, for

$$(x + iy) + (u + iv) = (x + u) + i(y + v) = (u + x) + i(v + y) = (u + iv) + (x + iy).$$

Associativity follows similarly. The 0 element is just $0 = 0 + 0i \in \mathbb{C}$. The additive inverse is $-x - iy$. We can prove distributivity by noting

$$(c + d)(x + iy) = (c + d)x + i(c + d)y.$$

Similarly for associativity. The scalar unit is $1 = 1 + 0i$.

The vector space of complex numbers is basically just $\mathbb{R}^2$ except you write a vector like $x + iy$ instead of (x, y). The complex numbers can be multiplied together so they have extra information though!

Solution (2.1.12). Assume a vector space has two 0 vectors, call them 0 and $0'$. Then by definition of a 0 element, we know that

$$0' + 0 = 0'.$$

But since $0'$ is also a zero element, then $0 + 0' = 0$ also. Since both are equal to $0 + 0'$, we conclude that $0 = 0'$, and the zero element is unique.

Solution (2.2.2abc). (a) This is not a subspace! In particular, the points $(-1, 0, 0)$ and $(0, -1, 0)$ satisfy the equation $x + y + z + 1 = 0$ but their sum does not. The sum is $(-1, -1, 0)$ and if you plug that into the equation, it is false. So this set is not closed under addition.

(b) This is a vector space. The set is the same as all multiples of the vector $(1, -1, 0)$. This is closed under addition, scalar multiplication, and has the 0 element. In particular, when $t = 0$, then $(0, 0, 0) \in W$. If we take two multiples, by say t_1 and t_2, adding them yields another multiple

$$t_1(1, -1, 0) + t_2(1, -1, 0) = (t_1 + t_2)(1, -1, 0) \in W.$$

Similarly, scaling gives $ct_1(1,-1,0) \in W$. Therefore all 3 properties of subspaces are satisfied, so W is a subspace.

(c) This is also a subspace. It is all vectors of the form

$$\begin{pmatrix} r-s \\ r+2s \\ -s \end{pmatrix} = r\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$$

which is closed under addition, scalar multiplication, and has the 0 element.

In particular when $r = s = 0$, then we get that $0 \in W$. Summing two vectors in W simplifies to

$$r_1\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + s_1\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} + r_2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + s_2\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} = (r_1+r_2)\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + (s_1+s_2)\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \in W.$$

Finally scaling by a constant c,

$$c\left(r\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}\right) = (cr)\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + (cs)\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \in W.$$

Therefore all 3 properties of subspaces are satisfied, so W is a subspace.