Textbook: 2.1.2, 2.2.22, 2.3.2, 2.3.3a, 2.3.23a, 2.3.31, 2.4.1, 2.4.3, 2.4.21

Solution (2.1.2). Here we have the somewhat strange formula that $(x, y) + (u, v) = (ux, vy)$ and $c(x, y) = (x^c, y^c)$. We show that this defines a vector space if we restrict our attention to (x, y) in the first quadrant, i.e. $x, y > 0$. So $V = \{(x, y) \mid x, y > 0\}$.

We have to show that these formulas for $+$ and $c(x, y)$ satisfy the 7 properties.

For commutivity,

$$(x, y) + (u, v) = (ux, vy) = (xu, yv) = (u, v) + (x, y).$$

For associativity,

$$(x, y) + ((u, v) + (w, z)) = (x, y) + (uw, vz) = (xuw, yvz) = (xu, yv) + (w, z) = ((x, y) + (u, v)) + (w, z).$$

The zero element is not $(0, 0)$ anymore. In fact $(0, 0) \notin V$ so it cannot be the zero vector. In fact now $\vec{0} = (1, 1)$ serves as the zero vector here. Remember the defining feature of the zero vector was that $\vec{v} + \vec{0} = \vec{v}$ no matter what v was. Here $\vec{0} = (1, 1)$ play that role. Indeed $(x, y) + (1, 1) = (1x, 1y) = (x, y)$.

Also, $-\vec{v}$ cannot be $(-x, -y)$ in this case. Things like $(-1, -2) \notin V$ anymore. We need $-\vec{v}$ to have the property that $v + (-\vec{v}) = \vec{0} = (1, 1)$. So in fact in this vector space

$$-(x, y) = (x^{-1}, y^{-1}) = \left(\frac{1}{x}, \frac{1}{y}\right).$$

Indeed $(x, y) + -(x, y) = (x, y) + (1/x, 1/y) = (x/x, y/y) = (1, 1) = \vec{0}$. Note that $(1/x, 1/y)$ exists since $x, y > 0$ and in particular $x, y \neq 0$.

Now we just have to show the other properties like distributivity and things like that. They are true as follows.

$$c(x, y) + c(u, v) = (x^c, y^c) + (u^c, v^c) = (x^c u^c, y^c v^c) = ((xu)^c, (yv)^c) = c((x, y) + (u, v))$$

$$(c + d)(x, y) = (x^{c+d}, y^{c+d}) = (x^c x^d, y^c y^d) = (x^c, y^c) + (x^d, y^d) = c(x, y) + d(x, y)$$

$$c(d(x, y)) = c(x^d, y^d) = ((x^d)^c, (y^d)^c) = (x^{cd}, y^{cd}) = (cd)(x, y)$$

Finally, the scalar $c = 1$ behaves trivially on V, by $1(x, y) = (x^1, y^1) = (x, y)$. Therefore we have shown all 7 properties of this weird $+$ and $\cdot$, so it is a vector space V.

Solution (2.2.22). (a) Let W and Z be subspaces of a vector space V. We show that their intersection $W \cap Z$ is also a subspace. Just to be clear, the intersection is

$$W \cap Z = \{v \in V \mid v \in W \text{ and } v \in Z\}$$

so $W \cap Z$ consists of vectors in both W and Z.

Well we just have to show that $W \cap Z$ satisfies the 3 properties of being a subspace, assuming W and Z both do. First, since W and Z are both subspace, they contain zero vector, i.e. $0 \in W$ and $0 \in Z$. So since 0 is in both, then $0 \in W \cap Z$.

Second, assume that $w_1, w_2 \in W \cap Z$. Then we need to show that $w_1 + w_2 \in W \cap Z$. Since $w_1, w_2 \in W \cap Z$, they are in both W and Z individually. Since both are subspace, they are closed under addition individually, so in particular $w_1 + w_2 \in W$ and $w_1 + w_2 \in Z$. Since $w_1 + w_2$ is in both, then $w_1 + w_2 \in W \cap Z$.

Third, given any scalar c and $\vec{w} \in W \cap Z$, we need to show that $c\vec{w} \in W \cap Z$ as well. Again since W and Z are subspace individually, then they are closed under scalar multiplication individually. So $w \in W$ and $w \in Z$ implies that $c\vec{w} \in W$ and $c\vec{w} \in Z$ for both. Since $c\vec{w}$ is in both, then $c\vec{w} \in W \cap Z$.

We have shown all 3 properties, so $W \cap Z$ is a subspace.

(b) The sum is also a subspace. I'll do this one a little faster. Define $W + Z = \{v \in V \mid v = w + z, w \in W, z \in Z\}$.

First, since $0 \in W$ and $0 \in Z$, then $0 = 0 + 0 \in W + Z$.

Second, given $w_1 + z_1 \in W + Z$ and $w_2 + z_2 \in W + Z$, then

$$w_1 + z_1 + w_2 + z_2 = (w_1 + w_2) + (z_1 + z_2).$$

Since $w_1 + w_2 \in W$ (remember W is a subspace), and $z_1 + z_2 \in Z$, then the total sum is in $W + Z$.

Finally, $c(w + z) = cw + cz$. Since $cw \in W$ and $cz \in Z$, then $c(w + z) \in W + Z$.

(c) Now we can prove that $W \cup Z$ is a subspace iff $Z \subseteq W$ or $W \subseteq Z$. Remember that $\cup$ refers to unioning, so $W \cup Z$ would be the set of vectors in either W OR Z.

Assume that $W \cup Z$ is a subspace. Then we have to show that either W was inside of Z or Z was inside of W to begin with. Suppose for contradiction that $W \not\subseteq Z$ and $Z \not\subseteq W$. Then there is a vector $w \in W$ but $w \notin Z$ and similarly let $z \in Z$ but $z \notin W$. Then $w + z \in W \cup Z$ since $W \cup Z$ is assumed to be a subspace. Let $v = w + z$. Since $v \in W \cup Z$, it is in either one or the other. Assume without loss of generality that $v \in W$. Then

$$v - w = (w + z) - w = z.$$

But $v \in W$ and $w \in W$ so $z = v - w \in W$. But we assumed that $z \notin W$ by construction. This is a contradiction, so either $Z \subseteq W$ or $W \subseteq Z$.

Conversely, assume $Z \subseteq W$ or $W \subseteq Z$. Then in either case $W \cup Z = W$ or Z (depending on which one's bigger), and it is a subspace since W and Z are.

Solution (2.3.2). To show that a vector lies in a span, we put the vectors as columns of a matrix and row reduce. In this case we row reduce the matrix as

$$\begin{pmatrix} 1 & -2 & -2 & -3 \\ -3 & 6 & 4 & 7 \\ -2 & 3 & 6 & 6 \\ 0 & 4 & -7 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Columns with leading 1's correspond to independent vectors, and else the column depends on the others. Therefore the last column depends on the other 3 and in particular we know what the relationship is from the entries. Indeed

$$3 \begin{pmatrix} 1 \\ -3 \\ -2 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} -2 \\ 6 \\ 3 \\ 4 \end{pmatrix} + 1 \begin{pmatrix} -2 \\ 4 \\ 6 \\ -7 \end{pmatrix} = \begin{pmatrix} -3 \\ 7 \\ 6 \\ 1 \end{pmatrix}.$$

Solution (2.3.31). (a) Suppose $v_1, \ldots, v_n$ are independent vectors, and we consider the subset of them $v_1, \ldots, v_k$ where $k < n$. We show this set is also independent. Indeed assume for contradiction that

$c_1v_1 + \dots c_kv_k = 0$ is a nontrivial linear dependence, i.e. at least one of the $c_i \neq 0$. But then we would have a nontrivial linear dependence between the original vectors

$$c_1v_1 + \dots c_kv_k + 0v_{k+1} + \dots 0v_n = 0.$$

Since $v_1, \dots, v_n$ were independent, then this is a contradiction. Thus $v_1, \dots, v_k$ are independent as well.

(b) This is not true for dependent vectors. Suppose we have $v_1 = (-1, 2)$ and $v_2 = (1, -2)$. These are dependent, but the subset $v_1 = (-1, 2)$ is independent.

Solution (2.4.1). All of these can be solved by putting the vectors into a matrix and row reducing. If you get the identity, it's a basis. If you don't, it's not a basis. Remember that you need 2 vectors to form a basis of $\mathbb{R}^2$, no more no less. So you don't even have to row reduce some of these.

(a) Yes (b) No (c) Yes (d) No (e) No.

Solution (2.4.3). Again you can find out everything you need to know by putting the vectors as columns of a matrix and row reducing to RREF. We get

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ -1 & -2 & -2 & 3 \\ 2 & 5 & 1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{pmatrix}.$$

(a) They do span $\mathbb{R}^3$ because we have 3 independent vectors. (b) They are not linearly independent since v_4 depends on the first 3. (c) They do not form a basis of $\mathbb{R}^3$. First of all, they're dependent. Second of all we have 4 of them and not 3 since $\dim \mathbb{R}^3 = 3$. (d) The dimension of the span is 3 since we have 3 independent vectors. The first 3 vectors form a basis.

Solution (2.4.21). (Proof 1) Suppose that $v_1, \dots, v_n$ form a basis of $\mathbb{R}^n$. Let A be a nonsingular (invertible) matrix. We show that $Av_1, \dots, Av_n$ is also a basis of $\mathbb{R}^n$.

Indeed we know that n vectors form a basis of $\mathbb{R}^n$ iff the matrix with the vectors as columns row reduces to the identity, which is the same as being invertible. So we need to show that the matrix

$$B = \begin{pmatrix} Av_1 & \dots & Av_n \end{pmatrix}$$

is invertible.

Let C be the matrix with v_i as columns. We know that C is invertible since the $v_1, \dots, v_n$ form a basis in the first place. Then by the matrix multiplication formula

$$B = \begin{pmatrix} Av_1 & \dots & Av_n \end{pmatrix} = AC.$$

Since both A and C are invertible, then so is B since $B^{-1} = C^{-1}A^{-1}$. Since B is invertible, its columns form basis.

(Proof 2) Since we have n vectors, $Av_1, \dots, Av_n$, then we just have to show that they are independent and they automatically span. We can show they are independent by definition. Suppose we have a linear relation

$$c_1Av_1 + \dots c_nAv_n = 0.$$

We show that $c_1 = c_2 = \dots = 0$ to show independence.

Factoring out A, we get that

$$A(c_1v_1 + \dots + c_nv_n) = 0$$

and multiplying both sides by A^{-1} yields that $c_1v_1 + \dots + c_nv_n = 0$. Since $v_1, \dots, v_n$ form a basis to begin with, then they are independent. Therefore by definition $c_i = 0$. Therefore $Av_1, \dots, Av_n$ are independent also. As noted, n independent vectors always form a basis of $\mathbb{R}^n$.