Textbook: 3.1.2abcd, 3.1.3, 3.1.21ab, 3.1.23ab, 3.1.26

3.1.2abcd (a) This is an inner product. It's an example of a weighted inner product from Example 3.3. We have bilinearity since

$$\begin{aligned}\langle c\vec{v}+d\vec{w},\vec{u}\rangle &= \langle (cv_1+dw_1, cv_2+dw_2),(u_1,u_2)\rangle \\ &= 2(cv_1+dw_1)u_1+3(cv_2+dw_2)u_2 \\ &= 2cv_1u_1+2dw_1u_1+3cv_2u_2+3dw_2u_2 \\ &= c(2v_1u_1+3v_2u_2)+d(2w_1u_1+3w_2u_2) \\ &= c\langle\vec{v},\vec{u}\rangle+d\langle\vec{w},\vec{u}\rangle\end{aligned}$$

and the second factor is the same. We have symmetry since

$$\langle v,w\rangle = 2v_1w_1+3v_2w_2 = 2w_1v_1+3w_2v_2 = \langle w,v\rangle.$$

Finally this is a positive form since $2,3>0$ and if $\vec{v}\neq 0$ then

$$\langle v,v\rangle = 2v_1^2+3v_2^2>0$$

since one of v_1 or v_2 is nonzero and squares are always positive. If $v=(0,0)$, then definitely $\|(0,0)\|=0$.

(b) This is not an inner product since we don't have positivity. If $\vec{v}=(-1,1)$, then

$$\langle v,v\rangle = (-1)(1)+(1)(-1)=-2.$$

(c) This is not an inner product because it is not positive. Suppose $\vec{v}=(-1,1)$ again. Then

$$\langle v,v\rangle = (-1+1)(-1+1)=0.$$

(d) This is not bilinear. In particular if $c=2$, $d=0$, v,w are arbitrary, then bilinearity reduces to showing that

$$\langle cv,w\rangle = c\langle v,w\rangle.$$

But we can see this is false for $\langle v,w\rangle = v_1^2w_1^2+v_2^2w_2^2$. In fact

$$\langle cv,w\rangle = \langle (cv_1,cv_2),(w_1,w_2)\rangle = (cv_1)^2w_1^2+(cv_2)^2w_2^2 = c^2(v_1^2w_1^2+v_2^2w_2^2) = c^2\langle v,w\rangle.$$

However you were supposed to get $c\langle v,w\rangle$, not $c^2\langle v,w\rangle$. In general these are different expressions (pick $c\neq 0,1$). So it is not bilinear and therefore not an inner product.

3.1.3 Actually now that I look at it, this is just the same as 3.1.2c factored out. This is not a positive form. Let $v=(-1,1)$. Then

$$\langle v,v\rangle = 1-1-1+1=0.$$

A nonzero vector cannot have 0 magnitude, so this is not an inner product.

3.1.21 (a)

$$\begin{aligned}\langle f,g\rangle &= \int_0^1 1x\,dx = \frac{1}{2} \\ \|f\| &= \sqrt{\int_0^1 1^2\,dx} = \sqrt{\int_0^1 1\,dx} = 1 \\ \|g\| &= \sqrt{\int_0^1 x^2\,dx} = \sqrt{\frac{1}{3}}\end{aligned}$$

(b)

$$\langle f, g\rangle = \int_0^1 \sin(2\pi x)\cos(2\pi x)\,dx = 0$$
$$\|f\| = \sqrt{\int_0^1 \sin(2\pi x)^2\,dx} = \sqrt{\frac{1}{2}}$$
$$\|g\| = \sqrt{\int_0^1 \cos(2\pi x)^2\,dx} = \sqrt{\frac{1}{2}}$$

3.1.23ab (a) This does define an inner product. First it is bilinear, similar to how the L^2 norm is bilinear.

$$\begin{aligned}\langle cf + dg, h\rangle &= \int_{-1}^1 (cf(x) + dg(x))h(x)e^{-x}\,dx \\ &= \int_{-1}^1 cf(x)h(x)e^{-x} + dg(x)h(x)e^{-x}\,dx \\ &= c\int_{-1}^1 f(x)h(x)e^{-x}\,dx + d\int_{-1}^1 g(x)h(x)e^{-x}\,dx \\ &= c\langle f, h\rangle + d\langle g, h\rangle\end{aligned}$$

The other equation for bilinear is the same. For symmetry, this is clear since

$$\langle f, g\rangle = \int_{-1}^1 f(x)g(x)e^{-x}\,dx = \int_{-1}^1 g(x)f(x)e^{-x}\,dx = \langle g, f\rangle.$$

Finally, remember that for all x, we know that $e^{-x} > 0$. Therefore, for any nonzero function $f(x)$, we know that $f(x)^2e^{-x} \geq 0$ and $f(x)e^{-x}$ has nonzero area under the curve. So we can conclude that

$$\langle f, f\rangle = \int_{-1}^1 f(x)^2e^{-x}\,dx > 0$$

so we have positivity. Therefore it is an inner product.

(b) This is not an inner product since it is not positive. Let $f(x) = -1$, the constant function. Then

$$\langle f, f\rangle = \int_{-1}^1 (-1)^2x\,dx = \int_{-1}^1 x\,dx = 0.$$

However $\langle f, f\rangle$ was supposed to be positive. Therefore this formula is not an inner product.

3.1.26 This is false. If we let $f(x) = x$ and $[a, b] = [0, 1]$, then on the one hand

$$\|f^2\| = \sqrt{\langle x^2, x^2\rangle} = \sqrt{\int_0^1 x^4\,dx} = \sqrt{\frac{1}{5}}.$$

But on the other hand

$$\|f\|^2 = \int_0^1 x\,dx = \frac{1}{2}.$$

Therefore $\|f^2\|$ and $\|f\|^2$ are not equal in general.