Problem 1. In this problem, we will walk through the proof why the L^1 norm $\|v\|_1 = \sum_{i=1}^{n} |v_i|$ on $\mathbb{R}^n$ has no associated inner product.

(a) Suppose we have an inner product $\langle -, - \rangle$ and its associated norm $\|v\| = \sqrt{\langle v, v \rangle}$. Show the equality

$$\langle v, w \rangle = \frac{1}{4}(\|v + w\|^2 - \|v - w\|^2).$$

This is called the polarization identity.

(b) Now suppose for contradiction that there is an inner product formula $\langle -, - \rangle_1$ such that $\|v\|_1 = \sqrt{\langle v, v \rangle_1}$. Then by a), we would know it had to have the formula $\langle v, w \rangle_1 = \frac{1}{4}(\|v + w\|_1^2 - \|v - w\|_1^2)$. But it turns out that this leads us to trouble. Show that the expression

$$\frac{1}{4}(\|v + w\|_1^2 - \|v - w\|_1^2)$$

is in fact NOT an inner product. This would complete the proof, by contradiction. (Hint: Maybe try showing that bilinearity on $\mathbb{R}^2$ is false.)

Solution. (a) Start on the right side and simplify to the left.

$$\begin{aligned}
\frac{1}{4}(\|v + w\|^2 - \|v - w\|^2) &= \frac{1}{4}\left(\langle v + w, v + w \rangle - \langle v - w, v - w \rangle\right) \\
&= \frac{1}{4}(\|v\|^2 + 2\langle v, w \rangle + \|w\|^2 - \|v\|^2 + 2\langle v, w \rangle - \|w\|^2) \\
&= \frac{1}{4}(4\langle v, w \rangle) = \langle v, w \rangle
\end{aligned}$$

(b) As in the problem statement, let

$$\langle v, w \rangle_1 = \frac{1}{4}(\|v + w\|_1^2 - \|v - w\|_1^2).$$

We can show that the expression is not an inner product. To make it easier, we can let $n = 2$ for now. Then let $\vec{u}, \vec{v} \in \mathbb{R}^2$. The expression written out in coordinates is

$$\langle u, v \rangle_1 = \frac{1}{4}((|u_1 + v_1| + |u_2 + v_2|)^2 - (|u_1 - v_1| + |u_2 - v_2|)^2).$$

It seems unlikely that this is bilinear. In particular, we should be able to pull out scalars, but this is in fact false. We claim that

$$\langle u, cv \rangle_1 \neq c\langle u, v \rangle_1.$$

To show that these are not equal, we can find one particular counterexample. Let $u = (1, 1)$, $v = (1, 0)$, and $c = 2$. Then on the one hand $\langle u, 2v \rangle_1 = \langle (1, 1), (2, 0) \rangle_1 = 0$, but on the other hand $2\langle u, v \rangle_1 = 2\langle (1, 1), (2, 0) \rangle_1 = 3$. Since $3 \neq 0$, we know that $\langle -, - \rangle_1$ is not bilinear and therefore not an inner product. If we want to generalize this counterexample to other n, one can just add on some zeroes to the end of these vectors. Consider $u = (1, 1, 0, \ldots, 0)$ and $v = (1, 0, 0, \ldots, 0)$. The calculations are the same.

Problem 2. Given an inner product $\langle -, - \rangle$, prove the parallelogram identity

$$2\|v\|^2 + 2\|w\|^2 = \|v + w\|^2 + \|v - w\|^2.$$

Solution. We can start with the right side and simplify to the left.

$$\begin{aligned}
\|v + w\|^2 + \|v - w\|^2 &= \langle v, v \rangle + 2\langle v, w \rangle + \langle w, w \rangle + \langle v, v \rangle - 2\langle v, w \rangle + \langle w, w \rangle \\
&= 2\langle v, v \rangle + 2\langle w, w \rangle \\
&= 2\|v\|^2 + 2\|w\|^2
\end{aligned}$$

Textbook: 3.2.9, 3.2.31, 3.2.36, 3.3.20abc, 3.4.1abcd, 3.4.10, 3.4.25

Solution (3.2.9). The Cauchy-Schwartz inequality is the fastest way to prove this inequality. Let $\vec{a} = (a_1, \ldots, a_n)$ and $\vec{1} = (1, 1, \ldots, 1)$. Then the Cauchy-Schwartz inequality says that

$$|\langle \vec{a}, \vec{1} \rangle| \leq \|\vec{a}\| \, \|\vec{1}\|.$$

Computing all the terms of this inequality out,

$$\begin{aligned} \langle \vec{a}, \vec{1} \rangle &= a_1 + \cdots + a_n \\ \|\vec{1}\| &= \sqrt{1 + \cdots + 1} = \sqrt{n} \\ \|\vec{a}\| &= \sqrt{a_1^2 + \cdots + a_n^2}. \end{aligned}$$

Plugging this into the C-S inequality and squaring both sides yields

$$(a_1 + \cdots + a_n)^2 \leq n(a_1^2 + \cdots + a_n^2).$$

Solution (3.2.36c). Let $p(x) = a + bx + cx^2$ be an orthogonal polynomial to 1 and x. We know that $\langle 1, p \rangle = 0$ and $\langle x, p \rangle = 0$, so we can turn these two equations into a system of linear equations in a, b, c.

$$\begin{aligned} \langle 1, p \rangle &= \int_{-1}^{1} a + bx + cx^2 \, dx = 2a + \frac{2}{3}c = 0 \\ \langle x, p \rangle &= \int_{-1}^{1} ax + bx^2 + cx^3 \, dx = \frac{2}{3}b = 0 \end{aligned}$$

Written as a matrix system, we obtain

$$\begin{pmatrix} 2 & 0 & 2/3 \\ 0 & 2/3 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Maybe this is some overkill, but the RREF is $\begin{pmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 0 \end{pmatrix}$ so we have a general solution

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -1/3 \\ 0 \\ 1 \end{pmatrix} c = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} c'.$$

Therefore polynomials of the form $p(x) = c(-1 + 3x^2)$ are all orthogonal to 1 and x

Solution (3.4.1abcd). (a) It is positive definite. The quadratic form is

$$q(x) = x^T K x = x_1^2 + 2x_2^2 > 0.$$

(b) It is not positive definite. The quadratic form is

$$q(x) = 3x_1 x_2$$

which is not positive, particularly if $x = (1, -1)$. OR it fails the determinant test.

(c) It is not positive definite. The quadratic form is

$$q(x) = x_1^2 + 4x_1x_2 + x_2^2.$$

In particular, this is not positive when $x = (-1, 1)$. Or you can argue it fails the determinant test.

(d) It is not positive definite. Again fails the determinant test. or you can see the quadratic form $q(x) = 5x_1^2 + 6x_1x_2 - 2x_2^2$ is negative for $x = (0, 1)$.

Solution (3.4.10). Let K be a nonsingular symmetric matrix. (a) If $x = Ky$ then

$$x^T K^{-1} x = (Ky)^T K^{-1} K y = y^T K^T K^{-1} K y = y^T K^T y = y^T K y.$$

This last step holds since $K^T = K$ is the definition of symmetric.

(b) If K is positive definite, then $y^T K y > 0$ for all $y \neq 0$. Now consider $x^T K^{-1} x$ where $x \neq 0$. Then since K is invertible, we have that $y = 0$ iff $x = Ky = 0$. Furthermore by part a),

$$x^T K^{-1} x = y^T K y > 0$$

for $x \neq 0$. Thus K^{-1} is positive definite.

Solution. The Gram matrix is

$$K = \begin{pmatrix} \langle 1, 1 \rangle & \langle 1, e^x \rangle & \langle 1, e^{2x} \rangle \\ \langle 1, e^x \rangle & \langle e^x, e^x \rangle & \langle e^x, e^{2x} \rangle \\ \langle 1, e^{2x} \rangle & \langle e^x, e^{2x} \rangle & \langle e^{2x}, e^{2x} \rangle \end{pmatrix} = \begin{pmatrix} 1 & e - 1 & \frac{1}{2}(e^2 - 1) \\ e - 1 & \frac{1}{2}(e^2 - 1) & \frac{1}{3}(e^3 - 1) \\ \frac{1}{2}(e^2 - 1) & \frac{1}{3}(e^3 - 1) & \frac{1}{4}(e^4 - 1) \end{pmatrix}.$$

This matrix is positive definite since the vectors were independent to begin with.