Textbook: 3.4.30b, 3.4.31, 3.6.26abc, 3.6.30b, 4.1.1abc, 4.1.4, 4.1.21, 4.1.23, 4.2.1

Hint for 3.4.31b: This problem is hard given what we've learned so far, so let me outline a solution.

- Prove that $\ker A \subseteq \ker A^T A$, i.e. $Ax = 0$ implies $A^T Ax = 0$.
- Prove that $\ker A^T A \subseteq \ker A$, i.e. $A^T Ax = 0$ implies $Ax = 0$.

  Start by showing that $Ax \in \operatorname{coker} A$ and $Ax \in \operatorname{img} A$

  Next use this to show that $\|Ax\|^2 = 0$.

  Conclude that $Ax = 0$.
- Conclude that $\ker A^T A = \ker A$.
- Use rank-nullity to show that $\operatorname{rank} A^T A = \operatorname{rank} A$.
- Use rank-nullity to show that $\operatorname{rank} AA^T = \operatorname{rank} A^T$.
- Conclude that $\operatorname{rank} A^T A = \operatorname{rank} AA^T$.

Solution (3.4.30b). Let $S = S^T$ be a symmetric and nonsingular matrix. We prove that S^2 is positive definite. (This is reminiscent of a^2 being positive for a nonzero real number a.)

We can show that for any $x \neq 0 \in \mathbb{R}^n$, that the quadratic for $x^T S^2 x > 0$. Indeed since S is symmetric:

$$x^T S^2 x = x^T S S x = x^T S^T S x = (Sx)^T (Sx) = \|Sx\|^2 .$$

Since S is nonsingular, then when $x \neq 0$, then $Sx \neq 0$ as well. By positivity of the norm, $Sx \neq 0$ implies that $\|Sx\|^2 > 0$ and the proof is complete.

Solution (3.4.31). (a) To see that $L = AA^T$ is a Gram matrix, we can write A in terms of its rows, as

$$A = \begin{pmatrix} \text{---} & a_{1*} & \text{---} \\ \text{---} & a_{2*} & \text{---} \\ & \vdots & \\ \text{---} & a_{m*} & \text{---} \end{pmatrix}.$$

Then compute L as

$$L = AA^T = \begin{pmatrix} \text{---} & a_{1*} & \text{---} \\ \text{---} & a_{2*} & \text{---} \\ & \vdots & \\ \text{---} & a_{m*} & \text{---} \end{pmatrix} \begin{pmatrix} | & | & & | \\ a_{1*} & a_{2*} & \dots & a_{m*} \\ | & | & & | \end{pmatrix} = \begin{pmatrix} a_{1*} \cdot a_{1*} & a_{1*} \cdot a_{2*} & \dots & a_{1*} \cdot a_{m*} \\ a_{1*} \cdot a_{2*} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{1*} \cdot a_{m*} & \dots & \dots & a_{m*} \cdot a_{m*} \end{pmatrix}.$$

Therefore L is the $m \times m$ Gram matrix of the rows of A with respect to the dot product on $\mathbb{R}^n$. Similarly $K = A^T A$ is the $n \times n$ Gram matrix of the columns of A.

(b) First, claim that $\ker A = \ker A^T A$. Indeed A is $m \times n$ and $A^T A$ is $n \times n$, so the kernels are both subspaces of $\mathbb{R}^n$. But even more than that, they are equal sets. First suppose that $x \in \ker A$, i.e. $Ax = 0$. Then $A^T Ax = A0 = 0$ so that $x \in \ker A^T A$. This shows that $\ker A \subseteq \ker A^T A$.

To show that $\ker A^T A \subseteq \ker A$, assume that $x \in \ker A^T A$, i.e. $A^T Ax = 0$. My hint is a little redundant, but $A^T (Ax) = 0$ means that $Ax \in \ker A^T = \operatorname{coker} A$ and $Ax \in \operatorname{img} A$ by definition. Anyhow, these equations help us show that $Ax = 0$. First,

$$\|Ax\|^2 = Ax \cdot Ax = (Ax)^T (Ax) = x^T A^T Ax = x^T (0) = 0.$$

Since $\|Ax\|^2 = 0$, the only possible vector Ax could be is 0. Therefore $Ax = 0$. Since $Ax = 0$, then $x \in \ker A$ and $\ker A = \ker A^T A$ as desired.

Finally we can finish this proof by rank-nullity. Since $\ker A = \ker A^T A$, then $\dim kerA = \dim \ker A^T A$. By rank-nullity

$$\operatorname{rank} A^T A = n - \dim \ker A^T A = n - \dim \ker A = \operatorname{rank} A.$$

Similarly

$$\operatorname{rank} AA^T = m - \dim \ker AA^T = m - \dim \ker A^T = \operatorname{rank} A^T.$$

But we know that $\operatorname{rank} A^T = \operatorname{rank} A$, so we can conclude that

$$\operatorname{rank} A^T A = \operatorname{rank} A = \operatorname{rank} A^T = \operatorname{rank} AA^T.$$

(c) We know that both $A^T A$ and AA^T are positive definite iff the columns and rows of A are both independent, by the main theorem of Gram matrices. However the only way for both the rows and columns to be independent is if the matrix A is square, and invertible by the fundamental theorem of linear algebra. Thus we need A to be square and invertible.

Solution (3.6.26). You can solve these by taking a determinant or by row reduction. I'll just say how they are dependent if they are dependent.

(a) Independent (b) Dependent $(1-i)\begin{pmatrix} 1+i \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1-i \end{pmatrix}$. (c) Independent

Solution (3.6.30b). Consider the matrix

$$\begin{pmatrix} 2 & -1+i & 1-2i \\ -4 & 3-i & 1+i \end{pmatrix}.$$

First, find the RREF form using the steps $r_2' = 2r_1 + r_2$, $r_2' = \frac{1}{1+i} r_2$, and $r_1' = (-1+i)r_2 + r_1$. Remember that

$$\frac{1}{1+i} = \frac{1}{1+i}\frac{1-i}{1-i} = \frac{1-i}{1^2+1^2} = \frac{1-i}{2}$$

so the last step is more simply $r_2' = \frac{1-i}{2} r_2$. Therefore the RREF is

$$\begin{pmatrix} 1 & 0 & -1-5i/2 \\ 0 & 1 & -3i \end{pmatrix}.$$

Then a basis for the kernel is

$$\ker A = \operatorname{span} \begin{pmatrix} 1+5i/2 \\ 3i \\ 1 \end{pmatrix}$$

and the image is

$$\operatorname{img} A = \operatorname{span} \begin{pmatrix} 2 \\ -4 \end{pmatrix}, \begin{pmatrix} -1+i \\ 3-i \end{pmatrix}.$$

We know that there are two independent columns, so there are two independent rows, namely all the rows are independent. So the rows form a basis of the coimage.

$$\operatorname{coimg} A = \operatorname{span} \begin{pmatrix} 2 \\ -1+i \\ 1-2i \end{pmatrix}, \begin{pmatrix} -4 \\ 3-i \\ 1+i \end{pmatrix}$$

Finally we know that $\operatorname{rank} A^T = 2$ and A^T has 2 columns, so rank-nullity says that $\dim \operatorname{coker} A = 2 - 2 = 0$. The only 0 dimensional subspace is $\operatorname{coker} A = 0$, so $v = 0$ is the basis.

Solution (4.1.1abc). (a) orthogonal basis (b) orthonormal basis (c) not a basis

Solution (4.1.4). It is clear that $\langle e_1, e_2\rangle = \langle e_1, e_3\rangle = \langle e_2, e_3\rangle = 0$, so the standard basis is orthogonal with respect to this weighted dot product. To find the orthonormal basis, we can calculate $u_1 = e_1/\|e_1\|$, etc. This yields

$$u_1 = (1,0,0) \quad u_2 = \left(0, \tfrac{1}{\sqrt{2}}, 0\right) \quad u_3 = \left(0, 0, \tfrac{1}{\sqrt{3}}\right).$$

Solution (4.1.23). (a) First, these two vectors are independent so they form a basis of $\mathbb{R}^2$. (Remember 2 independent vectors in $\mathbb{R}^2$ always span already.) They are orthogonal since

$$\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}2 & -1\\ -1 & 3\end{pmatrix}\begin{pmatrix}-2\\ 1\end{pmatrix} = \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}-2\\ 1\end{pmatrix} = 0.$$

(b) We can compute the coefficients as

$$c_1 = \frac{\langle v, v_1\rangle}{\|v_1\|^2} = \frac{\begin{pmatrix}3 & 2\end{pmatrix}\begin{pmatrix}2 & -1\\ -1 & 3\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix}}{\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}2 & -1\\ -1 & 3\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix}} = \frac{7}{3}$$

and

$$c_2 = \frac{\langle v, v_2\rangle}{\|v_2\|^2} = \frac{\begin{pmatrix}3 & 2\end{pmatrix}\begin{pmatrix}2 & -1\\ -1 & 3\end{pmatrix}\begin{pmatrix}-2\\ 1\end{pmatrix}}{\begin{pmatrix}-2 & 1\end{pmatrix}\begin{pmatrix}2 & -1\\ -1 & 3\end{pmatrix}\begin{pmatrix}-2\\ 1\end{pmatrix}} = \frac{-5}{15} = \frac{-1}{3}.$$

Therefore

$$\begin{pmatrix}3\\ 2\end{pmatrix} = \frac{7}{3}\begin{pmatrix}1\\ 1\end{pmatrix} + \frac{-1}{3}\begin{pmatrix}-2\\ 1\end{pmatrix}$$

which can see is true. It works without row reduction!

(c) According to (4.8), it should be that

$$\|(3,2)\|^2 = \left(\frac{7}{3}\right)^2 \|v_1\|^2 + \left(\frac{-1}{3}\right)^2 \|v_2\|^2 = \frac{49}{9}(3) + \frac{1}{9}(15) = 18.$$

Indeed

$$\|(3,2)\|^2 = \begin{pmatrix}3 & 2\end{pmatrix}\begin{pmatrix}2 & -1\\ -1 & 3\end{pmatrix}\begin{pmatrix}3\\ 2\end{pmatrix} = 18$$

as desired!

(d) To form an orthonormal basis, we can just divide by the magnitude of each basis vector.

$$u_1 = \tfrac{v_1}{\|v_1\|} = \tfrac{1}{\sqrt{3}}\begin{pmatrix}1\\ 1\end{pmatrix} \quad u_2 = \tfrac{v_2}{\|v_2\|} = \tfrac{1}{\sqrt{15}}\begin{pmatrix}-2\\ 1\end{pmatrix}.$$

(e) Now that we have an orthonormal basis we can use (4.5) instead. Indeed

$$\begin{pmatrix}3\\ 2\end{pmatrix} = \frac{7}{\sqrt{3}}u_1 + \frac{-5}{\sqrt{15}}u_2$$

as desired. Furthermore we can compute the norm

$$\|v\|^2 = \left(\frac{7}{\sqrt{3}}\right)^2 + \left(\frac{-5}{\sqrt{15}}\right)^2 = 18$$

which is also what we wanted!