Textbook: 4.2.8, 4.2.17ab, 4.3.1, 4.3.7, 4.3.11, 4.3.27bc, 4.4.3a

Solution (4.2.8b). For these problems, we can start with the vectors $e_1 = (1, 0)$ and $e_2 = (0, 1)$ and apply Gram-Schmidt for the various inner products. I'll do (b). The first step is short, $v_1 = e_1 = (1, 0)$. Then

$$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}}{\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{-1}{4} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

So an orthogonal basis is $(1, 0)$ and $(1/4, 1)$, so we can normalize them to find an orthonormal basis.

$$u_1 = \tfrac{v_1}{\|v_1\|} = \tfrac{1}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad u_2 = \tfrac{v_2}{\|v_2\|} = \tfrac{2}{\sqrt{3}} \begin{pmatrix} 1/4 \\ 1 \end{pmatrix}$$

Solution (4.3.1). (a) neither (b) proper orthogonal (c) only orthogonal (d) proper orthogonal (e) neither (f) proper orthogonal (g) only orthogonal

Solution (4.3.7). Let Q be orthogonal. Then Q^{-1} is also orthogonal since $Q^{-1} = Q^T$ in the first place! We already saw that Q^T was orthogonal, so Q^{-1} is also orthogonal as well.

Solution (4.3.11). We claim that an $n \times n$ orthogonal diagonal matrix must be of the form

$$\begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix}$$

i.e the diagonal entries are any choice of ± 1 (so there would be 2^n of them). Let Q be a diagonal orthogonal matrix. Then $Q^T Q = I$. But since Q is diagonal, then $Q^T = Q$ in the first place. Therefore $Q^2 = I$. This multiplying out Q^2, we see that $q_{ii}^2 = 1$ for all diagonal entries q_{ii}. Therefore $q_{ii} = \pm 1$, and any choice of $\pm$ works.