Textbook: 5.2.1, 5.2.3ad, 5.2.9, 5.3.1, 5.3.3

Solution (5.2.1). First we put the polynomial in $p(\vec{x}) = x^T Kx - 2x^T f + c$ form. This is

$$p(x) = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} - 2 \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -3/2 \end{pmatrix} + 2.$$

Therefore we know the minimum of this polynomial occurs at

$$x^* = K^{-1} f = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \\ -3/2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ -2 \end{pmatrix}.$$

The actual minimum value of the polynomial is

$$p(1/2, 1/2, -2) = c - (x^*)^T f = 2 - \begin{pmatrix} 1/2 & 1/2 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -3/2 \end{pmatrix} = -3/2.$$

Solution (5.2.9). Suppose we have a polynomial of the form $p(x) = x^T Kx - 2x^T f$, i.e. $c = 0$. We show that the actual minimum value is $p(x^*) \leq 0$.

Since $c = 0$ and the min value occurs at $x^* = K^{-1} f$, then

$$p(x^*) = c - (x^*)^T f = -(x^*)^T f = -(K^{-1} f)^T f = -f^T (K^{-1}) f.$$

(We used the fact that K^{-1} was symmetric here.) So it suffices to show that $f^T (K^{-1})^T f > 0$.

Note that in order for this polynomial to have a minimum at all, K must be positive definite. But remember that this means K^{-1} is positive definite (homework problem 3.4.10). Thus any quadratic form with K^{-1} in it will be positive. But $f^T K^{-1} f$ is a quadratic form with K^{-1}, so it is ≥ 0. Therefore $p(x^*) = -f^T K^{-1} f \leq 0$. In fact this is only 0 when $f = 0$, since K^{-1} is positive definite.