Textbook: 5.4.1bc, 5.5.1ab, 7.1.1, 7.1.2ab, 7.1.12, 7.1.19a-h, 7.2.24abc, 7.2.25ab, 2.6.4abe, 2.6.8, 2.6.10

Solution (5.5.1a). To find the best fit line $y = \beta t + \alpha$, calculate

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = (A^T A)^{-1} y$$

where

$$A = \begin{pmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 1 \\ 1 & 3 \end{pmatrix} \qquad y = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 5 \end{pmatrix}.$$

Then

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{52} \begin{pmatrix} 4 & 1 \\ 2 & 14 \end{pmatrix} \begin{pmatrix} 8 \\ 17 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 1 \end{pmatrix}.$$

So the best fit line is $y = t + 3/2$.

Solution (7.1.12). Let V be a vector space. Let $L : \mathbb{R} \to V$. If we view 1 as a vector with variable scalar x, then since we can pull scalars out of linear functions

$$L(x) = L(x \cdot 1) = xL(1).$$

Notice that $L(1)$ does not depend on the input x. Therefore we can call it a constant $L(1) = \vec{b}$. All in all, we have shown that $L(x) = x\vec{b}$ for some $\vec{b} \in V$.

Solution (7.1.19ag). (a) Let L be the operator $L(f) = f(0) + f(1)$. This is a function $L : C^1(\mathbb{R}) \to \mathbb{R}$ with codomain $\mathbb{R}$, since the set of possible outputs are real numbers. For example $L(x^2) = (0)^2 + (1)^2 = 1$, which is a real number. This is also a linear operator. To see this, verify the two properties of linear operators.

$$L(f+g) = (f+g)(0) + (f+g)(1) = f(0) + g(0) + f(1) + g(1) = f(0) + f(1) + g(0) + g(1) = L(f) + L(g)$$

$$L(cf) = (cf)(0) + (cf)(1) = cf(0) + cf(1) = c(f(0) + f(1)) = cL(f)$$

(g) Now let $L(f) = f(x) + 2$. This is a function $L : C^1(\mathbb{R}) \to C^1(\mathbb{R})$, with codomain also $C^(\mathbb{R})$. This is because the set of possible outputs $f(x) + 2$ are also differentiable functions. If f' exists, then $(f + 2)'$ exists, as by the sum rule it is equal to f' also. Therefore the output is in $C^1(\mathbb{R})$ and not $C^0(\mathbb{R})$.

But this function L is NOT linear. For example

$$L(cf) = (cf)(x) + 2 = cf(x) + 2 \neq cf(x) + 2c = c(f(x) + 2) = cL(f).$$

Solution (7.2.24c). Let $L(x, y) = (x - 4y, -2x + 3y)$. We will write this transformation in terms of the basis $v = (1, 1)$ and $w = (-1, 1)$. First we write L in terms of a matrix by pulling out the coordinates.

$$L \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -4 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Then the change of basis formula tells us how to write L in another basis. The formula says

$$B = S^{-1} A S = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & -4 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 2 & 5 \end{pmatrix}.$$

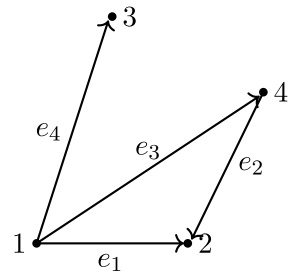
Therefore in v, w coordinates, L is given by the formula

$$L \begin{pmatrix} x \\ y \end{pmatrix}_{v,w} = \begin{pmatrix} -1 & 0 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}_{v,w}.$$

Solution (2.6.4a). This graph has 4 edges and 4 vertices, so the theorem states

$$\#\text{ind circ} = 1 - \#\text{vertices} + \#\text{edges} = 1 - 4 + 4 = 1.$$

Indeed see we can this visually since this graph has 1 "hole" in it. To be explicit, we need to label the edges and vertices of this graph. Any choice is fine, as long as you are consistent. I'll pick this one.



So we should get that the circuits are generated by $e_1 - e_2 - e_3$ (or maybe the negative of that). We can actually calculate this as follows. The boundary operator the linear transformation defined by $\partial(e_1) = v_2 - v_1$, $\partial(e_2) = v_2 - v_4$, $\partial(e_3) = v_4 - v_1$, and $\partial(e_4) = v_3 - v_1$. Putting this into matrix

$$M = \begin{pmatrix} -1 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{pmatrix}.$$

To find the circuits, we can compute the kernel of this matrix. Computing the RREF

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

we can see that the kernel is

$$\ker M = \begin{pmatrix} -z \\ z \\ z \\ 0 \end{pmatrix} = \text{span} \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

This gives the circuit $-e_1 + e_2 + e_3$ which is the negative of what we predicted. Furthermore, we did in fact verify the theorem, that there would be 1 independent cycle.

Solution (2.6.8). This problem is actually a little harder than I intended. If the graph G is connected, we can just use the vertex-edge formula. If G has n vertices and n edges, then we know that

$$1 - \#\text{ind circ} = n - n = 0$$

so we only have 1 independent circuit.

Actually the problem does not state that the graph is connected. Turns out to still be true if the graph has multiple connected components. So we have to prove this using a contradiction. Suppose G has multiple connected components $G_1, \ldots, G_m$, each with n_i vertices and k_i edges. Suppose for contradiction that G has no circuits whatsoever. Then any connected component G_i has no circuits either. Since G_i is connected, we can apply the formula, to note that $n_i = k_i + 1$, namely, each connected component has one more vertex than edges. But then in total

$$\#\text{vertices} = \sum_i n_i = \sum_i (k_i + 1) = \#\text{edges} + m > \#\text{edges}.$$

We were supposed to have the same number of edges and vertices! This is a contradiction, so G has at least 1 circuit somewhere.

Solution (2.6.10cd). (a) It took all my energy to make the graph from before, so I'll skip the drawings, but G_3 is a triangle, G_4 is a square with an X shape in the middle, and G_5 is a pentagon with a pentagram in the middle.

(b) These are large.

(c) We can compute the number of edges in G_n using a combinatorial argument. Since there is one edge for every pair of distinct vertices, then we need to count the number of pairs of distinct vertices. If there are n vertices, this is like putting the numbers 1 through n in a hat and picking out two of them. In this case we would n choices for the first number and $n-1$ choices for the second number. But $n(n-1)$ is double counting since the edges in the graph don't care about order, which the hat analogy does. So in total

$$\#\text{edges} = \frac{n(n-1)}{2}.$$

(d) Now we can apply the formula since G_n is connected.

$$\#\text{ind circ} = 1 - \#\text{vertices} + \#\text{edges} = 1 - n + \frac{n(n-1)}{2} = \frac{(n-1)(n-2)}{2}$$