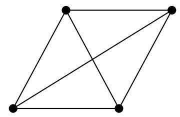
1. Consider the following graph.



(a) Use the vertex-edge formula to predict the number independent circuits for this graph. (b) Pick some labeling and edge arrows for this graph, and write down some independent circuits explicitly. (c) Use the boundary operator to find a basis of circuits for this graph explicitly. Don't worry I won't have you row reduce a giant matrix on the exam. (d) Does your answer in part (c) jive with part (a)?

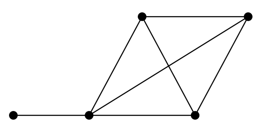
Solution. First, the dimension of the cokernel is the number of independent circuits in the We know that by the Euler characteristic formula

$$\dim \operatorname{coker}(A) = 1 - \text{number of vertices} + \text{number of edges} = 1 - 4 + 6 = 3.$$

Visually, the three independent circuits are the equalateral triangle in the bottom left, and the two longer triangles down the middle. The fourth equilateral triangle is a linear combination of the other three circuits.

I finished the rest of the problem by hand at the end of the document.

2. Consider the graph from problem 1 but with one edge attached to it. (It looks like a kite!)



Explain both visually and using the vertex-edge formula why this graph has the same number of independent circuits as the graph in problem 1.

Solution. This graph has the same number of independent circuits because it has the same number of "holes". Adding a little tail won't change that. More mathematically, we added one vertex and one edge, so now we have 5 vertices and 7 edges. In any event their difference $v - e$ is still -2, so that we have 3 independent circuits.

3. Consider the system $Ax = b$ where

$$A = \begin{pmatrix} 0 & 1 \\ -3 & 1 \\ 2 & 2 \end{pmatrix} \quad b = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}.$$

(a) Find the least squares solution to this system.

(b) Which element $w^* \in \operatorname{img}(A)$ actually is at minimum distance from b?

Solution. (a) The least squares solution is

$$x^* = (A^TA)^{-1}A^Tb = \frac{1}{77}\begin{pmatrix} 6 & -1 \\ -1 & 13 \end{pmatrix}\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \frac{1}{77}\begin{pmatrix} 1 \\ -13 \end{pmatrix}.$$

(b) The actual closest point from the image of A to B is

$$w^* = Ax^* = \frac{1}{77}\begin{pmatrix} -13 \\ -16 \\ -24 \end{pmatrix}.$$

4. Consider the matrix

$$A = \frac{1}{3}\begin{pmatrix} 1 & 1 & -1 \\ -1 & 2 & 0 \\ -1 & 2 & 0 \end{pmatrix}.$$

(a) Without even doing any calculation, you should be able to look at this matrix and know one of the eigenvalues. What is that eigenvalue and why?

(b) Compute the other eigenvalues of A and diagonalize it.

Solution. (a) This matrix has two rows which are exactly the same. Therefore the rows are dependent. Therefore the rank of A^T is less than 3, which means the rank of A is less than 3. The main theorem for invertible matrices tells us that therefore the $\ker A \neq 0$. Thus $\lambda = 0$ is eigenvalue.

(b) The characteristic polynomial is

$$\det(A - \lambda I) = -\lambda^3 + 3\lambda^2 - 2\lambda = -\lambda(\lambda - 1)(\lambda - 2) = 0.$$

So we have $\lambda = 0, 1, 2$ as the eigenvalues. Finding the eigenvectors is a matter of solving $(A - \lambda I)v = 0$ as usual. Putting it all together the diagonalization is

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{pmatrix}^{(} - 1).$$

5. Find the eigenvalues and eigensapces of the matrix

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 0 & -2 & -3 \\ 0 & 1 & 2 \end{pmatrix}.$$

What are the algebraic and geometric multiplicities of each eigenvalue?

Solution. To diagonalize a matrix, we just need the eigenvalues and a basis of eigenvectors. The eigenvalues are $\lambda = -1, 1, 1$ with eigenvectors $v_1 = (2, -3, 1)$, $v_2 = (0, -1, 1)$, and $v_3 = (1, 0, 0)$. The eigenvalue $\lambda = -1$ has algebraic and geometric multiplicities both 1. Then the algebraic and geometric multiplicities of $\lambda = 1$ are both 2, since it repeats twice in the characteristic polynomial and also has 2 eigenvectors.

6. Compute e^B where

$$B = \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}.$$

Solution. First we find the Jordan decomposition. The eigenvalues are $\lambda = 0$ and $\lambda = -4$ with eigenvectors $v_1 = (1, 2)$ and $v_2 = (-1, 2)$. Therefore

$$\begin{aligned}
e^{\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}} &= \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix} e^{\begin{pmatrix} 0 & 0 \\ 0 & -4 \end{pmatrix}} \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}^{-1} \\
&= \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} e^0 & 0 \\ 0 & e^{-4} \end{pmatrix} \frac{1}{4} \begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1/2 + 1/2e^4 & 1/4 - 1/4e^4 \\ 1 - 1/e^4 & 1/2 + 1/2e^4 \end{pmatrix}
\end{aligned}$$

7. Find the Jordan decomposition of the matrix

$$C = \begin{pmatrix} 2 & -1 & 0 \\ 9 & -4 & -3 \\ 0 & 0 & -1 \end{pmatrix}.$$

Use the Jordan decomposition to compute e^C.

Solution. Finding the eigenvalues, we get that we have a triple eigenvalue $\lambda = -1$. But the only eigenvector is $v = (1, 3, 0)$. So we need to generalized eigenvectors to complete the Jordan chain v, w_1, w_2. First, we can find w_1 by solving

$$(C - (-1)I)w_1 = v.$$

Solving this system gives a solution

$$w_1 = \frac{y}{3} \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + \begin{pmatrix} 1/3 \\ 0 \\ 0 \end{pmatrix}.$$

We don't want vectors already in the kernel, so we pick $w_1 = (1/3, 0, 0)$. Now w_2 completes the chain, so we solve $(C - (-1)I)w_2 = w_1$, which gives us

$$w_2 = \frac{y}{3} \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + \begin{pmatrix} 1/9 \\ 0 \\ 1/3 \end{pmatrix}.$$

So we can pick $w_2 = (1/9, 0, 1/3)$. Therefore the Jordan decomposition is $C = SJS^{-1}$ where

$$S = \begin{pmatrix} 1 & 1/3 & 1/9 \\ 3 & 0 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}$$

and

$$J = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

Now we can use this to compute the matrix exponential e^C. By the Jordan form we know that

$$e^C = Se^J S^{-1} = Se^\Lambda e^N S^{-1} = Se^\Lambda(I + N + \frac{1}{2}N^2)S^{-1}.$$

Written out this is

$$e^C = S \begin{pmatrix} e^{-1} & 0 & 0 \\ 0 & e^{-1} & 0 \\ 0 & 0 & e^{-1} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} S^{-1} = \begin{pmatrix} 4/e & -1/e & 3/2e \\ 9/e & -2/e & 3/2e \\ 0 & 0 & 1/e \end{pmatrix}.$$

8. Let

$$D = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -1 & 0 \\ 3 & 0 & 1 \end{pmatrix}.$$

(a) Find an orthonormal basis of eigenvectors for D.

(b) Use that orthonormal basis to write down the spectral decomposition of D.

(c) Exam 2 Review: Compute the QR decomposition of D

Solution. This is a symmetric matrix, so we can make an orthonormal basis of eigenvectors. This is the spectral decomposition. Find the eigenvectors normally, we get that $\lambda = -1, -2, 4$ with eigenvectors $v_1 = (0, 1, 0), v_2 = (1, 0, -1), v_3 = (1, 0, 1)$. Making these unit vectors we get that the orthogonal change of basis matrix is

$$Q = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

and

$$\Lambda = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

The spectral decomposition is $D = Q\Lambda Q^T$.

The QR decomposition is different. To calculate it, you do alternate G-S on the columns of D, so the resulting orthonormal basis forms Q and the coefficients from G-S make R. Remember that alternate G-S relies on the recursive process. Let $w_1 = (1, 0, 3)$, $w_2 = (0, -1, 0)$, and $w_3 = (3, 0, 1)$. The basis of unit vectors we will get at the end is u_1, u_2, u_3. First, $w_1 = r_{11}w_1$, so that $r_{11} = \|w_1\| = \sqrt{10}$ and $u_1 = \frac{1}{\sqrt{10}}(1, 0, 3)$.

Then $w_2 = r_{12}u_1 + r_{22}u_2$. We know that $r_{12} = w_2 \cdot u_1 = 0$ and therefore $u_2 = w_2$ since w_2 is already a unit vector orthogonal to u_1. We saw that $r_{12} = 0$ and $r_{22} = 1$.

Now onto $w_3 = r_{13}u_1 + r_{23}u_2 + r_{33}u_3$. First $r_{13} = w_3 \cdot u_1 = \frac{6}{\sqrt{10}}$, and $r_{23} = w_3 \cdot u_2 = 0$. Finally

$$r_{33} = \sqrt{\|w_3\|^2 - r_{13}^2 - r_{23}^2} = \sqrt{10 - 36/10} = \sqrt{64/10} = \frac{8}{\sqrt{10}}.$$

Finally

$$u_3 = \frac{w_3 - r_{13}u_1 - r_{23}u_2}{r_{33}} = \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}.$$

Therefore the QR decomposition is

$$D = \begin{pmatrix} \frac{1}{\sqrt{10}} & 0 & \frac{3}{\sqrt{10}} \\ 0 & -1 & 0 \\ \frac{3}{\sqrt{10}} & 0 & -\frac{1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} \sqrt{10} & 0 & \frac{6}{\sqrt{10}} \\ 0 & 1 & 0 \\ 0 & 0 & \frac{8}{\sqrt{10}} \end{pmatrix}.$$

9. Let B be a positive definite symmetric matrix. Suppose B^2 has spectral decomposition

$$B^2 = Q\Lambda Q^T.$$

Find a spectral decomposition of B in terms of Q and Λ.

Solution. Let Λ have diagonal entries λ_i, i.e. the eigenvalues of B^2. Let μ_i be the eigenvalues of B. Since B is positive definite then it has all positive eigenvalues, so $\mu_i > 0$. But the eigenvalues of B^2 are the eigenvalues of B, but squared, so $\mu_i^2 = \lambda_i$ with the same eigenvectors. So the change of basis matrix Q is still the same, but the matrix of eigenvalues is the positive square root of what it used to be, $\mu_i = +\sqrt{\lambda_i}$. We can denote the diagonal matrix with the μ_i as $\sqrt{\Lambda}$. Therefore

$$B = Q\sqrt{\Lambda}Q^T.$$

10. Suppose A is a square matrix with two different diagonalizations

$$S\Lambda S^{-1} = A = T\Lambda' T^{-1}.$$

Do Λ and Λ' have to be equal matrices? If not, what do they have in common? What about S and T?

Solution. Neither the diagonal entries nor the change of basis matrices need be the same. While the eigenvalues have to be the same, we can rearrange the order of them. So Λ and Λ' are the same, but the eigenvalues can be in a different order. If we rearrange the eigenvalues, then we also need to rearrange the eigenvectors, so S and T could have their columns permuted. We can also change S by scaling any eigenvector, or in general changing the basis of eigenvectors for any eigenspace V_λ. Therefore S and T definitely can be different.

Try finding two different diagonalizations of the matrix from Problem 5.

11. Does the quadratic

$$q(x, y, z) = 3x^2 + 2xy + 3y^2 + 2z^2 - 3x + 2y - 4z - 2$$

have a minimum value? If so, find the vector (x^*, y^*, z^*) which achieves the minumum, and calculate the actual minimum value.

Solution. Writing this in matrix form, we know that $q(\vec{x}) = x^T K x - 2x^T f - 2$ where

$$K = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} 3/2 \\ -1 \\ 2 \end{pmatrix}.$$

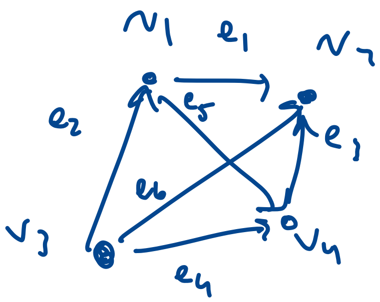
By the formula, this only has a minimum when K is positive definite. Calculating the eigenvalues show that $\lambda = 2, 2, 4 > 0$ so K is positive definite. Then the minimizer is $x^* = K^{-1} f = (11/16, -9/16, 1)$. The actual minimum value is $q(x^*) = 2 - x^{*T} f = -179/32$.

12. Write the following transformation in terms of the given basis β of $\mathbb{R}^3$.

$$L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x - 3y + z \\ -y + z \\ -x + y \end{pmatrix} \qquad \beta = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

Solution. The matrix corresponding to this transformation is $A = \begin{pmatrix} 2 & -3 & 1 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$. So therefore L in terms of the basis β is represented by the matrix $B = S^{-1} A S$ where S is a matrix with β as the columns. In total

$$B = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 & -3 & 1 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$



We predict the 3 independent circuits to be

- $e_1 - e_6 + e_2 = c_1$
- $e_4 + e_3 - e_6 = c_2$
- $e_2 + e_5 - e_4 = c_3$

The last circuit $e_1 - e_3 - e_5$ is

$c_1 - c_2 - c_3$, so not independent.

To calculate them explicitly $\partial(e) = v_{end} - v_{start}$ so

$$\partial = M = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{pmatrix} -1 & 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

The circuits form a basis of the kernel of M

$\ker M = \operatorname{Span}\left(\begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right)$

* $-e_1 - e_2 + e_3 + e_4 = -c_1 + c_2$
* $-e_1 + e_3 + e_5 = -c_1 + c_2 + c_3$
* $-e_1 - e_2 + e_6 = -c_1$

So our predicted basis did jive w/ what M gave us. Since they are linear combos of the circuits we guessed.

- $e_1 - e_6 + e_2 = c_1$
- $e_4 + e_3 - e_6 = c_2$
- $e_2 + e_5 - e_4 = c_3$