1. Consider the following two vector functions $x_1(t) = \begin{bmatrix} 1 \\ t \end{bmatrix}$ and $x_2(t) = \begin{bmatrix} t \\ t^2 \end{bmatrix}$. (a) Calculate $W(x_1, x_2)$. (b) Show that these two vector functions are independent. (c) Explain why parts (a) and (b) don't contradict each other.

Solution: (a) $W(x_1, x_2) = \det \begin{bmatrix} 1 & t \\ t & t^2 \end{bmatrix} = (1)(t^2) - (t)^2 = 0$ (b) They are independent since the equation $c_1 \begin{bmatrix} 1 \\ t \end{bmatrix} + c_2 \begin{bmatrix} t \\ t^2 \end{bmatrix} = 0$ for all t, forces $c_1 = c_2 = 0$. Plugging in $t = 0$, gives $c_1 = 0$. Then $c_2 \begin{bmatrix} t \\ t^2 \end{bmatrix} = 0$ which means $c_2 = 0$. (c) They don't contradict each other since x_1 and x_2 are no solutions to a linear differential equation.

2. Solve the following differential equation, and find the domain of the solution. Must the solution be unique on that domain?

$$y' = \frac{y}{x^2 - 1} \qquad y(0) = 2$$

Solution: $y = \frac{2\sqrt{|x-1|}}{\sqrt{|x+1|}}$, the domain is $x \neq -1$. This solution is unique on an interval $I \subset (-1, 1)$ since the equation has discontinuities as $x = \pm 1$, by the theorem in 1.3.

3. Consider a population of rabbits with a birth rate $\beta = 0.5P$ thousand births per time and a death rate of $\delta = 0.1$ thousand deaths per time. Find the differential equation that models this population and determine the carrying/threshold capacity. Draw the phase plane for this model and determine the stability of the critical points.

Solution: The model is $P' = P(\beta - \delta) = P(0.5P - 0.1) = -0.5P(0.2 - P)$, so we have an explosion/extinction model. The threshold capacity is $M = 0.2 = 200$ rabbits. The phase plane has an unstable equilibrium at $P = 2$ and a stable on at $P = 0$.

4. Draw the bifurcation diagram of the system

$$y' = yh^2 - y^2$$

and identify any possible bifurcation points.

Solution: The picture of the bifurcation diagram is a parabola $y = h^2$ and a straight line $y = 0$.

Note: I think I will update the solutions for #3 and #4 with pictures tomorrow afternoon.

5. Show that if B is a noninvertible square matrix, then B has an eigenvalue $\lambda = 0$.

Solution: If B is noninvertible, then the big theorem of linear algebra says that $Bv = 0$ does not have a unique solution. Reworded, the solution space of B is nontrivial. Therefore there is a nonzero vector v in the solutions space of B. So $Bv = 0 = 0v$. Thus by definition $\lambda = 0$ is an eigenvalue of B.

Alternatively, we know $\det B = 0 = \lambda_1 \ldots \lambda_n$. Since the product is 0, we must have one of the $\lambda = 0$.

6. Find the set of vectors $\vec{b}$ such that the following system has an infinite number of solutions.

$$\begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \vec{b}$$

What the vectors $\vec{b}$ where there is a unique solution? What about no solutions?

Solution: There is never a unique solution. Let $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. Then if $b_1 + b_2 = 0$, then the system has infinite solutions. If $b_1 + b_2 \neq 0$, the system is inconsistent and there are no solutions.

7. Let A be a matrix with solution space W. Let B be a matrix, with solution space U, and such that you can row reduce A to B. How does the solution space U compare to W?

Solution: Row reduction preserves the solution space of a matrix. Therefore $U = W$.

8. Show that $\{(0,1,2,0),(1,1,0,0),(-1,2,3,-1),(-3,0,0,0)\}$ is a basis of $\mathbb{R}^4$.

Solution: Put the vectors into the columns of a matrix, and take the determinant.

$$\det \begin{bmatrix} 0 & 1 & -1 & -3 \\ 1 & 1 & 2 & 0 \\ 2 & 0 & 3 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} = 6 \neq 0$$

Since the determinant is nonzero, then the columns are independent. In general, 4 independent vectors in $\mathbb{R}^4$ form a basis, so that our vectors must form a basis.

9. Let W be the set of vectors $(x,y,z,w) \in \mathbb{R}^4$ such that $x + y = z + w$ and $3x - 2z = 0$ form a subspace. Find a matrix A such that W is the solution space of A.

Solution: $A = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 3 & 0 & -2 & 0 \end{bmatrix}$

10. Find the general solutions to the following differential equations.

i. $y''' + 4y'' + 4y' = x^2$

ii. $y'' + y = \sin(x)$

iii. $y' + \sin(x)y = \sin(x)$

Solution:

i. $y = c_1 e^{-2x} + c_2 x e^{-2x} + c_3 + \frac{1}{12}x^3 - \frac{1}{4}x^2 + \frac{9}{24}x$

ii. $y = c_1 \cos(x) + c_2 \sin(x) - \frac{1}{2}x\cos(x)$

iii. $y = c_1 e^{\cos(x)} + 1$

11. Calculate the determinant of the following matrix, and determine whether it is invertible or not.

$$\begin{bmatrix} -1 & 2 & -2 & 1 & -2 \\ 3 & 2 & -1 & 2 & 2 \\ 0 & 2 & -1 & 1 & -2 \\ 3 & 2 & 1 & 2 & 2 \\ 1 & 1 & 4 & -2 & 1 \end{bmatrix}$$

Solution: We row reduce this matrix a little to make the determinant easier to compute. Because adding a multiple of one row to another doesn't change the determinant. The 2nd and 4th rows are different by one number, so we can substract $R_4 - R_2$ to make the 4th row have four 0's. Expand by the new 4th row. Then to compute the resulting 4×4, notice that the 1st and 3rd rows are off by 1 number. Do the same trick again, and replace R_1 by $R_1 - R_3$. Now you can expand by the first row, and you only have a 3×3 to compute. The answer turns out to be $\det A = -48$, so A is invertible.

12. Find the amplitude and time lag for the function $x(t)$ that satisfies

$$x'' + 4x = 0$$

with $x(0) = 1$ and $x'(0) = 1$.

Solution: $\omega_0 = 2$, $C = \sqrt{5}/2$, $\delta = 0.232$.

13. Show that the functions $e^{r_1 x}$ and $e^{r_2 x}$ are independent when $r_1 \neq r_2$.

Solution: These equations are solutions to a linear differential equation, so we can apply the Wronksian.

$$\det \begin{bmatrix} e^{r_1 x} & e^{r_2 x} \\ r_1 e^{r_1 x} & r_2 e^{r_2 x} \end{bmatrix} = (r_2 - r_1)e^{(r_1 + r_2)x} \neq 0$$

Since $r_2 - r_1 \neq 0$, then the Wronskian is nonzero and they are independent.

14. Show that the piecewise function

$$y(x) = \begin{cases} 0 & x < c \\ (x-c)^2 & x \geq c \end{cases}$$

is a solution to the differential equation $y' = 2\sqrt{y}$. Is the piecewise function when $c = 2$ independent from the function $y = x^2$?

Solution: This piecewise function is a solution to the differential equation since it solves the equation for both $x < c$ and $x \geq c$. Indeed $y = 0$ is a solution and so is $y = (x-c)^2$. This function is continuous and differentialable since the $y = 0$ and $y = (x-c)^2$ line up perfectly at $x = c$. An argument similar to #1 shows that when $c = 2$, the piecewise function is independent from $y = x^2$. (Let me know if you have questions on this one, it's long to write up.) We cannot apply the Wronskian since we don't have a linear differential equation.

This equation $y' = \sqrt{y}$ doesn't have unique solutions in a neighborhood of $x = 0$, and these piecewise functions show that in fact it it has infinite solutions for the initial value $y(0) = 0$. Each piecewise function for $c > 0$ satisfies the differential equation and the IV $y(0) = 0$.

15. Consider the differential equation $y' = 1$. Is $y = |x|$ a solution to this differential equation on all of $\mathbb{R}$?

Solution: $y = |x|$ is not a solution on all of $\mathbb{R}$, since the derivative for negative numbers is $y' = -1$. However it is a solution if we restrict to the interval $(0, \infty)$, but then it is equal to $y = x$.

16. Solve the linear system

$$\vec{x}' = \begin{bmatrix} -4 & -1 & 1 \\ 2 & 0 & 0 \\ -7 & -2 & 2 \end{bmatrix} \vec{x}.$$

Solution: $\vec{x} = c_1 e^{-t} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + c_2 e^{-t} \left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix} \right) + c_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

17. Solve the linear system

$$\vec{x}' = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -4 & -7 \\ -1 & 3 & 5 \end{bmatrix} \vec{x}$$

Solution: $\vec{x} = c_1 \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} + c_2 \left(\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} t + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right) + c_3 \left(\frac{1}{2} \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} t^2 + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} -3 \\ -1 \\ 0 \end{bmatrix} \right)$

18. Find the type and stability of the critical points for the following almost linear system. Use the table on page 520 if you need. Make a sketch of the solutions near each critical point, which you should do without looking in the book.

$$\begin{aligned} x' &= x + xy + y + 1 \\ y' &= x^2 + y^2 - 10 \end{aligned}$$

Solution: The critical points are $(\pm 1, \pm 3)$. The linearization matrix and the nature of the critical point is listen below.

i. $(-1, 3)$, $\begin{bmatrix} 4 & 0 \\ -2 & 6 \end{bmatrix}$, unstable improper node

ii. $(-1, -3)$, $\begin{bmatrix} -2 & 0 \\ -2 & -6 \end{bmatrix}$, stable improper node

iii. $(3, -1)$, $\begin{bmatrix} 0 & 4 \\ 6 & -2 \end{bmatrix}$, saddle point

iv. $(-3, -1)$, $\begin{bmatrix} 0 & -2 \\ -6 & -2 \end{bmatrix}$, saddle point

19. Calculate the solutions to the following two IVP's using the Laplace transform.

i. $y'' - y = \cos(2x) \qquad y(0) = 2 \quad y'(0) = -1$

ii. $y' - y = t^4 \qquad y(0) = 1$

Solution:

i. $y = \frac{8}{5} e^{-x} + \frac{3}{5} e^{x} - \frac{1}{5} \cos(2x)$

ii. $y = 25e^t - t^4 - 4t^3 - 12t^2 = 24t = 24$

20. For the following linear system, is the critical point at the origin asymptotically stable, stable, or unstable?

$$\begin{aligned} x' &= -x + y \\ y' &= -3x + y \end{aligned}$$

Solution: You get the origin is a center, since both eigenvalues are imaginary. Centers are stable points, but not asymptotically stable.

21. Let W be the set of continuous functions f such that $\int_0^1 f(t)\,dt = 0$. Show that W is a subspace of the vector space of continuous functions. Is it possible to find a finite set of functions $\{f_1, \dots, f_n\}$ that form a basis of W?

Solution: Let $f, g \in W$. Then

$$\int_0^1 f(t) + g(t)\,dt = \int_0^1 f(t)\,dt + \int_0^1 g(t)\,dt = 0 + 0 = 0$$

so $f + g \in W$. Similarly if a is a scalar, then

$$\int_0^1 af(t)\,dt = 0$$

as well, so $af \in W$.

There is no finite basis $\{f_1, \dots, f_n\}$ for W. You can find an infinite amount of independent functions in W. For example $(2t-1)^{2k+1}$ is a function in W for all $k \geq 0$. So $2t-1$, $(2t-1)^3$, $(2t-1)^5$, etc, are all in W, and they are independent. You can compute the integral as follows.

$$\int_0^1 (2t-1)^{2k+1}\,dt = \left(\frac{1}{2(2k+2)}(2t-1)^{2k+2}\right)\Bigg|_0^1 = \frac{1}{2(2k+2)}\left((1)^{2k+2} - (-1)^{2k+2}\right) = 0.$$

You get zero at the end since $(-1)^{2k+2} = (1)^{2k+2}$, which is because $2k+2$ is even. You can show these functions are independent since they are polynomials with increasing degrees.

22. Consider two subspaces W_1 and W_2 of $\mathbb{R}^4$. We have seen on Midterm 2 that the intersection $W_1 \cap W_2$ also forms a subspace of $\mathbb{R}^4$. If $\dim W_1 = 2$ and $\dim W_2 = 2$, what are the possible values for $\dim(W_1 \cap W_2)$? For an extra challenge, find examples of each possible dimension.

Solution: The possible dimensions for $W_1 \cap W_2$ are 0,1, and 2. An example when $\dim(W_1 \cap W_2) = 0$ (i.e. when the subspace itself is 0) is

$$W_1 = \{(x_1, x_2, 0, 0)\} \qquad W_2 = \{(0, 0, x_3, x_4)\}.$$

The two subspaces only have vector $\vec{0}$ in common, so $W_1 \cap W_2 = 0$. For dim $= 1$, you can use

$$W_1 = \{(x_1, x_2, 0, 0)\} \qquad W_2 = \{(x_1, 0, x_3, 0)\}.$$

Then $W_1 \cap W_2 = \{(x_1, 0, 0, 0)\}$ which is 1-dimensional. For dim $= 2$, you can pick any 2-dimensional subspace W, and let $W = W_1 = W_2$. Thus $\dim(W_1 \cap W_2) = \dim W = 2$.

23. Consider 4 vector functions $\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4$ from $\mathbb{R} \to \mathbb{R}^3$. Namely, these vector functions have 3 output functions, like $\begin{bmatrix} e^t \\ e^t \\ 2e^t \end{bmatrix}$ for example. Is it possible for $\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4$ to be independent, or must they be dependent?

Side Note: Contrast this with how 4 normal vectors in $\mathbb{R}^3$ can never be independent!

Solution: It is possible for 4 vector functions with 3 outputs to be independent. For example consider the following functions.

$$x_1 = \begin{bmatrix} e^t \\ 0 \\ 0 \end{bmatrix} \qquad x_2 = \begin{bmatrix} e^{2t} \\ 0 \\ 0 \end{bmatrix} \qquad x_3 = \begin{bmatrix} e^{3t} \\ 0 \\ 0 \end{bmatrix} \qquad x_4 = \begin{bmatrix} e^{4t} \\ 0 \\ 0 \end{bmatrix}.$$

Since e^t, e^{2t}, e^{3t}, e^{4t} are all independent functions, then the above vector functions are independent as well.

24. Can you find a 2×2 matrix A such that $A^2 = I$? Is there more than one possibility for the matrix A? What are the possible eigenvalues of A? (This problem shows that matrices do not have unique square roots.)

Solution: Some examples are $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ or $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. Both of these matrices have the property $A^2 = I$. So there is more than one possibility. In fact, there are an infinite amount of possibilities. Any matrix similar to A also has this property. For example, let $B = PAP^{-1}$ and $A^2 = I$. Then

$$B^2 = PA^2P^{-1} = PIP^{-1} = PP^{-1} = I.$$

Another way to say this is the the identity I has an infinite amount of square roots.

In general, if $A^2 = I$, then the eigenvalues of A must be $\lambda = \pm 1$. Here is the proof. If λ is an eigenvalue of A, then $Av = \lambda v$. Multiplying both sides by A, you get $Iv = \lambda^2 v$. So λ^2 is an eigenvalue of I. But only eigenvalue of the identity I is one. Therefore $\lambda^2 = 1$, and so $\lambda = \pm 1$. We see from the above examples that you can get either $\lambda = 1$ of $\lambda = -1$, so both possibilities for the eigenvalues are achieved.

25. Find the general solution for the following differential equation.

$$y^{(3)} + y'' + y' = x^3$$

Solution: $y = \frac{1}{4}x^4 - x^3 + 6x + c_1 + e^{-x/2}(c_2 \cos(\sqrt{3}x/2) + c_2 \sin(\sqrt{3}x/2))$