Homework 10 Solutions
February 1, 2020

11.15.1. Find $\iint_S x\cos(x+y)\,dx\,dy$ where S is the triangular region with vertices $(0,0)$, $(\pi,0)$, and (π,π).

Solution. The region S is bounded by the functions $y=0$ and $y=x$ between $x=0$ and $x=\pi$. Then the integral is

$$\iint_S x\cos(x+y)\,dx\,dy = \int_0^\pi \int_0^x x\cos(x+y)\,dx\,dy = -3\pi/2.$$

11.15.4. Find $\iint_S x^2y^2\,dx\,dy$ where S is the bounded portion of the first quadrant lying between the two hyperbolas $xy=1$ and $xy=2$, and two straight lines $y=x$ and $y=4x$.

Solution. This region is some kind of warped square which is neither type 1 nor type 2, so we break it up into separate integrals. Find the corners of the shape, and use the x components in order to find the regions.

$$\begin{aligned}&\iint_S x^2y^2\,dx\,dy\\ &= \int_{1/2}^{1/\sqrt{2}} \int_{1/x}^{4x} x^2y^2\,dx\,dy + \int_{1/\sqrt{2}}^{1} \int_{1/x}^{2/x} x^2y^2\,dx\,dy + \int_1^{\sqrt{2}} \int_x^{2/x} x^2y^2\,dx\,dy\\ &= \frac{7\ln(2)}{3}\end{aligned}$$

11.15.5. Find $\iint_S x^2-y^2\,dx\,dy$ where S is bounded by the curve $y=\sin x$ and the interval $[0,\pi]$.

Solution. We have

$$\iint_S x^2-y^2\,dx\,dy = \int_0^\pi \int_0^{\sin x} x^2-y^2\,dy\,dx = \pi^2 - 40/9.$$

11.15.7. A solid is bounded by the surface $z=x^2-y^2$, the xy-plane, and the planes $x=1$ and $x=3$. Make a sketch of the solid and compute its volume by double integration.

Solution. Let me know if you need a sketch in person. The region in the xy-plane formed by the solid is bounded between $y=-x$ and $y=x$, and between $x=1$ and $x=3$. So the integral is

$$\int_1^3 \int_{-x}^{x} x^2-y^2\,dy\,dx = 80/3.$$

11.15.11,14. Interchange the order of integration, and assume such a process works. Draw a sketch of the region. (11) $\int_1^4 \int_{\sqrt{x}}^2 f\, dy\, dx$ (14) $\int_1^e \int_0^{\int} \log x f\, dy\, dx$

Solution. (11) The region is bounded by a sideways parabola and a line, and you only look at the part between $x = 1$ and $x = 4$.

$$\int_1^4 \int_{\sqrt{x}}^2 f\, dy\, dx = \int_1^2 \int_1^{y^2} f\, dx\, dy$$

(14) The region is bounded by $y = 0$ and $y = \ln(x)$, in between $x = 1$ and $x = e$. This is when $\ln(x)$ hits $y = 0$ and $y = 1$.

$$\int_1^e \int_0^{\log x} f\, dy\, dx = \int_0^1 \int_{e^y}^e f\, dx\, dy$$

11.18.2. Sketch the following region and determine the centroid. S is bounded by $y^2 = x + 3$ and $y^2 = 5 - x$.

Solution. The shape of S is a oval thing with 2 corners which is bounded by 2 parabolas. The area is

$$\iint_S 1\, dA = \int_{-1}^1 \int_{y^2-3}^{5-y^2} 1\, dx\, dy = 8/3.$$

Then

$$\overline{x} = \frac{3}{8} \iint_S x\, dA = 4$$

and

$$\overline{y} = \frac{3}{8} \iint_S y\, dA = 0.$$

11.22.1ad. Use Green's theorem to evaluate the line integral $\int_C y^2\, dx + x\, dy$ when (a) C is the square $[0, 2] \times [0, 2]$ and (d) C is the circle of radius 2 and center at the origin.

Solution. (a) By Green's theorem

$$\int_C y^2\, dx + x\, dy = \int_0^2 \int_0^2 1 - 2y\, dx\, dy = -4.$$

(d) By Green's theorem and symmetry about the x-axis,

$$\int_C y^2\, dx + x\, dy = \iint_{B_2(0)} 1 - 2y\, dx\, dy = 0.$$

11.22.2. If $P(x,y) = xe^{-y^2}$ and $Q(x,y) = -x^2ye^{-y^2} + \frac{1}{x^2+y^2}$, evaluate the line integral $\int_C P\,dx + Q\,dy$ where C is the boundary of the square $[-a,a]\times[-a,a]$.

Solution. Let $F = (P,Q)$. Note that

$$F = \nabla(x^2e^{-y^2})/2 + \left(0, \frac{1}{x^2+y^2}\right).$$

Thus

$$\int_C F \cdot ds = 0 + \int_C \left(0, \frac{1}{x^2+y^2}\right) \cdot ds$$

since conservative vector fields integrate to 0 around closed curves. Doing the second integral manually, yields

$$\int_C \left(0, \frac{1}{x^2+y^2}\right) \cdot ds = 0$$

because integrating one side is the negative of the integral of the other side, if you write it out. Thus the total integral is 0.

11.22.4. Given two scalar fields u and v that are continuously differentiable on an open set containing the circular disk R whose boundary is the circle $x^2+y^2=1$, define two vector fields f, g by $f = (v,u)$ and $g = (u_x - u_y, v_x - v_y)$. Find $\iint_R f \cdot g\,dx\,dy$ if it is known that on the boundary of R we have $u = 1$ and $v = y$.

Solution. By Green's theorem:

$$\begin{aligned}
\iint_R f \cdot g\,dx\,dy &= \iint vu_x - vu_y + uv_x - uv_y\,dx\,dy \\
&= \iint_R (uv)_x - (uv)_y\,dx\,dy \\
&= \int_{\partial R} (uv, uv) \cdot ds \\
&= \int_{\partial R} (y, y) \cdot ds \\
&= \int_0^{2\pi} (\sin(t), \sin(t)) \cdot (-\sin(t), \cos(t))\,dt \\
&= -\pi.
\end{aligned}$$

11.22.5. If f and g are continuously differentiable in an open connected set S in the plane, show that $\int_C f\nabla g \cdot ds = -\int_C g\nabla f \cdot ds$

Solution. By definitions, product rule, and Green's theorem:

$$\begin{aligned}
\int_C f\nabla g + g\nabla f \cdot ds &= \int_C (fg_x + gf_x, fg_y + gf_y) \cdot ds \\
&= \int_C ((fg)_x, (fg)_y) \cdot ds \\
&= \iint_R (fg)_{xy} - (fg)_{xy}\, dx\, dy = 0
\end{aligned}$$

11.22.7. If $f = (Q, -P)$, show that

$$\int_C P\,dx + Q\,dy = \int_C f \cdot n\, ds.$$

Solution. By definitions:

$$\begin{aligned}
\int_C f \cdot n\, ds &= \int_C f(\alpha(t)) \cdot n(t)||\alpha'(t)||\, dt \\
&= \int_C f(\alpha(t)) \cdot (y'(t), -x'(t))\, dt \\
&= \int_C Q(\alpha(t))y'(t) + P(\alpha(t))x'(t)\, dt \\
&= \int_C P\,dx + Q\,dy.
\end{aligned}$$

11.22.8ab. Let f and g be scalar fields with continuous first and second order partial derivatives on an open set S in the plane. Let R denote a region in S whose boundary is a piecewise smooth Jordan curve C. Prove the following identities. (a) $\int_C \nabla g \cdot n\, ds = \iint_R \nabla^2 g\, dx\, dy$ (b) $\int_C f(\nabla g \cdot n)\, ds = \iint_R f\nabla^2 g + \nabla f \cdot \nabla g\, dx\, dy$

Solution. (a) By definitions, the previous problem, and Green's theorem:

$$\begin{aligned}
\int_C \nabla g \cdot n\, ds &= \int_C (g_x, g_y) \cdot n\, ds \\
&= \int_C -g_y\, dx + g_x\, dy \\
&= \iint_R g_{xx} + g_{yy}\, dx\, dy.
\end{aligned}$$

(b) By definitions, the previous problem, and Green's theorem:

$$\begin{aligned}\int_C f(\nabla g \cdot n)\, ds &= \int_C (fg_x, fg_y) \cdot n\, ds \\ &= \int_C -fg_y\, dx + fg_x\, dy \\ &= \iint_R f(g_{xx} + g_{yy}) + f_x g_x + f_y g_y\, dx\, dy.\end{aligned}$$

11.25.3. A connected plane region with exactly one hole is called doubly connected. If P and Q are continuously differentiable on an open doubly connected region R and if $P_y = Q_x$ everywhere in R, how many distinct values are possible for line integrals $\int_C P\, dx + Q\, dy$ taken around piecewise smooth Jordan curves in R?

Solution. If the curve C does not go around the hole, then $\int_C (P, Q) \cdot ds = 0$ by Green's theorem. If C is any simple closed curve going around the hole, we claim that C only goes around the hole once. Else, the intermediate value theorem implies that the curve crosses itself somewhere. (Note: This is wildly unrigorous, but given that we haven't defined what a hole is, I'll leave it unrigorous.)

So let $I = \int_C P\, dx + Q\, dy$. We claim that the integral around any other curve in the same direction is also I. (Otherwise it is just $-I$.).

Take any two simple Jordan curves C_1, C_2 in R. In order to prove that the integrals around these curves are the same, we need to apply Green's theorem. However the theorem proved in the book holds only for when C_2 is contained in the interior of the region inside of C_1. This isn't true for two general curves (they can intersect in any number of ways), so we have to make a C_3 inside C_2 and C_1 and then show that the integrals around C_1 and C_2 are both the same as that of C_3. This will be our strategy.

So we claim that there is a simple Jordan curve C_3 contained in the interior of both C_1 and C_2. Since R is open, there exists an open ball around every point in R also contained in R. Assume the hole in R contains the origin WLOG. Define C_1' to be the curve defined as follows. If $\alpha_1 : [0, 1] \to \mathbb{R}^2$ define $\tilde{\alpha}_1 : [0, 1] \to \mathbb{R}^2$ by $\tilde{\alpha}_1(t) = (1 - \varepsilon)\alpha(t)$. Since R is open and $[0, 1]$ is closed and bounded, we can find a fixed ε small enough such that $\tilde{\alpha}_1(t) \in R$. (I believe this is by something called the Lebesgue number lemma.) Do the same process to form $\tilde{\alpha}_2$. Then take α_3 to be the curve

$$\alpha_3(t) = \begin{cases} \tilde{\alpha}_1(t) & |\tilde{\alpha}_1(t)| \leq |\tilde{\alpha}_2(t)| \\ \tilde{\alpha}_2(t) & |\tilde{\alpha}_2'(t)| \leq |\tilde{\alpha}_1(t)| \end{cases}.$$

This is a piecewise Jordan curve in general and is continuous when α_1 and α_2 go around in a circle at the same rate, which we can assume WLOG (if not, just change the speeds). Then α_3 defines the curve C_3 in the interior of C_1 and C_2. (Actually I think this doesn't hold if α_1 makes a zig-zag shape but you get the idea.)

Long story short, since R is open, we can find a curve C_3 inside of C_1 and C_2. Then Green's theorem in the book implies that $\int_{C_1-C_3} P\,dx + Q\,dy = 0$ and $\int_{C_2-C_3} P\,dx + Q\,dy = 0$. Therefore

$$\int_{C_1} P\,dx + Q\,dy = \int_{C_3} P\,dx + Q\,dy = \int_{C_2} P\,dx + Q\,dy.$$

So in total we can have up to 3 values; 0, I, and $-I$ (assuming $I \neq 0$).

11.25.4. Solve 11.25.3 for triply connected regions, that is, for connected plane regions with exactly 2 holes.

Solution. I'm not going to go through the same argument as before where I show that the curves agree, I'm just going to list all the cases. First, if the curve goes around no holes, then the integral is 0. If it goes around one hole, then it will be some number $\pm I_1$, and likewise for $\pm I_2$. If it goes around both holes, since it is simple, then it cannot cross itself so it goes around the holes in the same direction so we get $\pm(I_1 + I_2)$. So we can have up to 7 possible values.

11.25.5ab. Let P and Q be two scalar fields which have continuous derivatives satisfying $P_y = Q_x$ everywhere in the plane except at three points. Let C_1, C_2, and C_3 be nonintersecting circles centered at these points. Let $I_k = \int_{C_k} P\,dx + Q\,dy$. Assume that $I_1 = 12$, $I_2 = 10$, and $I_3 = 15$. (a) Find the integral around a figure 8 C containing C_1 and C_3. (b) Draw another closed curve Γ along which $\int_\Gamma P\,dx + Q\,dy = 1$.

Solution. (a) Let α be the curve $C - C_1 + C_3$. By Green's theorem

$$\int_\alpha P\,dx + Q\,dy = 0.$$

But therefore

$$\int_C P\,dx + Q\,dy = I_1 - I_3 = 12 - 15 = -3.$$

(b) Note that $(3)(15)-(2)(10)-(2)(12) = 1$. Let Γ be a curve that goes around C_1 2 times clockwise, goes around C_2 2 times clockwise, and goes around C_3 3 times counterclockwise. Then integrating $\Gamma + 2C_1 + 2C_2 - 3C_3$ will yield 0 by Green's theorem. But then

$$\int_\Gamma P\,dx + Q\,dy = -2I_1 - 2I_2 + 3I_3 = 1$$

as desired.