Homework 11 Solutions
February 1, 2020

11.28.3. Express the integral over the region $\{(x, y) \mid a^2 \leq x^2 + y^2 \leq b^2\} \subset \mathbb{R}^2$ as a iterated integral in polar coordinates.

Solution. This region is an annulus with inner radius a and outer radius b. Name the region S. Then

$$\iint_S 1\, dA = \int_0^{2\pi} \int_a^b r\, dr\, d\theta.$$

11.28.8. Transform the integral $\int_0^1 \int_{x^2}^{x} (x^2 + y^2)^{-1/2}\, dy\, dx$ into polar and compute the value.

Solution. The region is below the line at angle $\pi/4$ and the radial component is bounded by $y = x^2$, which has polar equation $r \sin\theta = r^2 \cos^2\theta$, so that $r = \tan\theta / \cos theta$. Then the integral is

$$\int_0^{\pi/4} \int_0^{\tan\theta/\cos\theta} r/r\, dr\, d\theta = \int_0^{\pi/4} \frac{\tan\theta}{\cos\theta}\, d\theta = \sqrt{2} - 1.$$

11.28.9. Repeat the exercise above for $\int_0^a \int_0^{\sqrt{a^2-y^2}} x^2 + y^2\, dx\, dy$.

Solution. The region in question is the first quadrant of a disc of radius a centered at the origin. In polar this is

$$\int_0^{\pi/2} \int_0^a r^2\, dr\, d\theta = \int_0^{\pi/2} \frac{a^3}{3}\, d\theta = \frac{a^4\pi}{8}.$$

11.28.14. Find a suitable linear transformation to compute

$$\iint_S (x - y)^2 \sin^2(x + y)\, dx\, dy$$

where S is the parallelogram $(\pi, 0)$, $(2\pi, \pi)$, $(\pi, 2\pi)$, and $(0, \pi)$.

Solution. Pick $u = x + y$ and $v = x - y$. Inverting this linear transformation, we see that $x = (u + v)/2$ and $y = (u - v)/2$. Thus $J(u, v) = 1/2$ and the inverse image of the parallegram is the square bounded by $u = \pi$, $u = 3\pi$, $v = -\pi$, and $v = \pi$. Then the integral is

$$\begin{aligned} \iint_S (x - y)^2 \sin^2(x + y)\, dx\, dy &= \int_{-\pi}^{\pi} \int_{\pi}^{3\pi} v^2 \sin^2(u)\, du\, dv \\ &= \frac{2\pi^4}{3}. \end{aligned}$$

11.28.16abc. Let $r > 0$, and let $I(r) = \int_{-r}^{r} e^{-u^2}\, du$. (a) Show that $I^2(r) = \iint_R e^{-(x^2+y^2)}\, dx\, dy$ where $R = [-r, r]^2$. (b) If C_1 and C_2 are circular disks inscribing and circumscribing R, show that the integrals on these discs bound $I^2(r)$. (c) Express the integrals over C_1 and C_2 in polar coordinates and use (b) to deduce that $I(r) \to \sqrt{\pi}$ as $r \to \infty$.

Solution. (a) Since e^{-u^2} is smooth on $\mathbb{R}$, then the product of the integrals is the iterated integral of the product of the integrands.

$$\left(\int_{-r}^{r} e^{-u^2}\, du\right)^2 = \left(\int_{-r}^{r} e^{-x^2}\, dx\right)\left(\int_{-r}^{r} e^{-y^2}\, dy\right) = \iint_R e^{-(x^2+y^2)}\, dx\, dy.$$

(b) Let T be a bounded region containing C_1, R, and C_2. Let $f_S(x, y) = e^{-(x^2+y^2)}\chi_S(x, y)$, where

$$\chi_S(x, y) = \begin{cases} 1 & (x, y) \in S \\ 0 & \text{else} \end{cases}$$

and S is any set. Then $\iint_T f_R(x, y)\, dx\, dy = \iint_R f(x, y)\, dx\, dy$ and similarly for C_1 and C_2. But note that $f_{C_1} \leq f_R \leq f_{C_2}$ on T. Since integrating positive functions preserves inequalities, then integrals on C_1 and C_2 bound that of R as desired.

(c) By the squeeze theorem and part (a), it suffices to show that

$$\lim_{r\to\infty} \iint_{C_r} f(x, y)\, dx\, dy = \pi.$$

Indeed transforming the integral into polar coordinates shows that

$$\begin{aligned} \lim_{r\to\infty} \iint_{C_r} f(x, y)\, dx\, dy &= \lim_{r\to\infty} \int_0^{2\pi} \int_0^{r} e^{-r_0^2} r_0\, dr_0\, d\theta \\ &= 2\pi \lim_{r\to\infty} \left(-\frac{1}{2}e^{-r^2} + \frac{1}{2}\right) = \pi \end{aligned}$$

since $e^{-x^2} \to 0$ as $x \to \infty$.