Homework 12 Solutions
February 1, 2020

11.34.1,4. Evaluate two triple integrals. (1) $\iiint_S xy^2z^3\,dx\,dy\,dz$ where S is the solid bounded by $z = xy$, $y = x$, $x = 1$, and $z = 0$. (4) $\iiint_S \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$ where S is the region bounded by the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

Solution. (1) The integral can be evaluated as follows.

$$\begin{aligned}\iiint_S xy^2z^3\,dx\,dy\,dz &= \int_0^1\int_0^x\int_0^{xy} xy^2z^3\,dz\,dy\,dx\\ &= \int_0^1\int_0^x \frac{1}{4}x^5y^6\,dy\,dx\\ &= \frac{1}{28}\int_0^1 x^{13}\,dx = \frac{1}{364}\end{aligned}$$

(4) We use a change of variables $x = au$, $y = bv$, $z = cw$. This has Jacobian abc, and the preimage of the region under the transformation is the unit sphere. Then we can write the integral in spherical coordaintes. Therefore by the change of variables theorem:

$$\begin{aligned}\iiint_S \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\,dx\,dy\,dz &= abc\iiint_{B(1)} u^2 + v^2 + w^2\,du\,dv\,dw\\ &= abc\int_0^\pi\int_0^{2\pi}\int_0^1 \rho^2(\rho^2\sin\varphi)\,d\rho\,d\theta\,d\varphi\\ &= 2abc\pi\left(\int_0^1\rho^4\,d\rho\right)\left(\int_0^\pi\sin\varphi\,d\varphi\right) = \frac{4abc\pi}{5}\end{aligned}$$

11.34.6,8. For the following integrals, change the order of integration and describe the region of integration S. (6) $\int_0^1\int_0^{1-x}\int_0^{x+y} dz\,dy\,dx$ and
(8) $\int_0^1\int_0^1\int_0^{x^2+y^2} 1\,dx\,dy\,dx$

Solution. (6) The region is the graph of the plane $z = x + y$ over the triangle formed by $x = 0$, $y = 0$, and $y = 1 - x$ in the xy-plane. We can rearrange the integral as follows, for example.

$$\int_0^1\int_0^{1-x}\int_0^{x+y} dz\,dy\,dx = \int_0^1\int_z^1\int_0^{x-z} 1\,dy\,dx\,dz$$

(8) The region is bounded by the paraboloid $z = x^2 + y^2$ and is above unit square. In order to change the order of integration, we need multiple integrals since the bounding function of the region is piecewise with respect to x or y.

$$\int_0^1 \int_0^1 \int_0^{x^2+y^2} 1\,dx\,dy\,dx = \int_0^1 \int_0^{x^2} \int_0^1 1\,dy\,dz\,dx + \int_0^1 \int_{x^2}^1 \int_{\sqrt{z-x^2}}^1 1\,dy\,dz\,dx$$

11.34.9. Show that

$$\int_0^x \int_0^v \int_0^u f(t)\,dt\,du\,dv = \frac{1}{2}\int_0^x (x-t)^2 f(t)\,dt.$$

Solution. The integral is over a region bounded by the plane $t = u$, and above the triangle bounded by $u = v$, $v = 0$, and $u = c$ in the uv-plane. Changing the order of integration to $dv\,du\,dt$, we obtain the following.

$$\begin{aligned}\int_0^x \int_0^v \int_0^u f(t)\,dt\,du\,dv &= \int_0^x \int_t^x \int_u^x f(t)\,dv\,du\,dt \\ &= \int_0^x \int_t^x f(t)x - f(t)u\,du\,dt \\ &= \int_0^x \frac{1}{2}f(t)x^2 - f(t)xt + \frac{1}{2}f(t)t^2\,dt \\ &= \frac{1}{2}\int_0^x f(t)(x-t)^2\,dt\end{aligned}$$

11.34.10. Evaluate the following integral by transforming it into cylindrical coordinates: $\iiint_S x^2 + y^2\,dx\,dy\,dz$ where S is the solid bounded by the surface $x^2 + y^2 = 2z$ and the plane $z = 2$.

Solution. The region is above the paraboloid $z = (x^2+y^2)/2$ and below $z = 2$. This gives the z-bounds. The projection of the region to the xy-plane is a circle of radius 2. Therefore the transformation is

$$\begin{aligned}\iiint_S x^2 + y^2\,dx\,dy\,dz &= \int_0^{2\pi} \int_0^2 \int_{r^2/2}^2 r^2(r)\,dz\,dr\,d\theta \\ &= \pi \int_0^4 2r^3 - \frac{1}{2}r^5\,dr = \frac{16\pi}{3}.\end{aligned}$$

11.34.15. Transform the following integral into spherical coordinates:

$$\iiint_S ((x-a)^2 + (y-b)^2 + (z-c)^2)^{-1/2}\,dx\,dy\,dz$$

where S is a sphere of radius R centered at the origin, and (a, b, c) it outside the sphere.

Solution. By changing basis using a rotation matrix, it suffices to let $(a,b,c) = (0,0,\lambda)$, where $\lambda = ||(a,b,c)||$. Note that a rotation matrix has determinant 1, so that the Jacobian contributes no scaling factor.

Now by transformation of spherical coordinates:

$$\begin{aligned}
&\iiint_S (x^2+y^2+(z-\lambda)^2)^{-1/2}\,dx\,dy\,dz \\
&\qquad = \int_0^{2\pi}\int_0^{\pi}\int_0^R \frac{\rho^2\sin(\varphi)}{\sqrt{\rho^2-2\rho\lambda\cos(\varphi)+\lambda^2}}\,d\rho\,d\varphi\,d\theta \\
&\qquad = 2\pi\int_0^R\int_{-1}^1 \frac{\rho^2}{\sqrt{\rho^2+\lambda^2+2\rho\lambda u}}\,du\,d\rho \\
&\qquad = 2\pi\int_0^R \frac{\rho}{2\lambda}\left(\sqrt{\lambda^2+2\lambda\rho+\rho^2}-\sqrt{\lambda^2-2\lambda\rho+\rho^2}\right)d\rho \\
&\qquad = 4\pi\int_0^R \frac{\rho^2}{\lambda}\,d\rho = \frac{4\pi R^3}{3\lambda}.
\end{aligned}$$

11.34.31. Let $S_n(a)$ denote the set of points with $\sum_i |x_i| \le a$. Denote the volume by $V_n(a) = \int_{S_n(a)} 1\,dx_1\ldots dx_n$. (a) Prove that $V_n(a) = a^n V_n(1)$. (b) For $n \ge 2$, express the integral $V_n(1)$ as an iteration of a one dimensional integral and an $n-1$ integral and show that

$$V_n(1) = V_{n-1}(1)\int_{-1}^1 (1-|x|)^{n-1}\,dx = \frac{2}{n}V_{n-1}(1).$$

(c) Use the previous parts to deduce that $V_n(a) = \frac{2^n a^n}{n!}$.

Solution. (a) Consider the change of variables $u_i = ax_i$. The Jacobian is a^n and the preimage of $S_n(a)$ is $S_n(1)$. Therefore

$$V_n(a) = \int_{S_n(a)} 1\,du_1\ldots du_n = a^n\int_{S_n(1)} 1\,dx_1\ldots dx_n = a^n V_n(1).$$

(b) Taking a constant value of x_1, the cross section of $S_n(1)$ is $S_{n-1}(1-|x_1|)$ since we can rearrange the equation

$$|x_2| + \cdots + |x_n| \le 1 - |x_1|.$$

Doing the $x_2,\ldots,x_n$ integrals first, the iterated volume integral becomes

$$V_n(1) = \int_{-1}^1 V_{n-1}(1-|x_1|)\,dx_1 = \int_{-1}^1 V_{n-1}(1)(1-|x_1|)^{n-1}\,dx_1.$$

(c) Now we compute $\int_{-1}^{1}(1-|x|)^{n-1}\,dx$. By breaking up the integral over two intervals:

$$\int_{-1}^{1}(1-|x|)^{n-1}\,dx = \int_{-1}^{0}(1+x)^{n-1}\,dx + \int_{0}^{1}(1-x)^{n-1}\,dx = \frac{2}{n}.$$

Therefore we have the recursive formaula $V_n(1) = \frac{2}{n}V_{n-1}(1)$. Since $V_1(1) = 2$, then $V_n(1) = \frac{2^n}{n!}$, so that $V_n(a) = \frac{2^n a^n}{n!}$. Notice that $\lim_{n\to\infty} V_n(a) = 0$ since $n!$ grows faster than $(2a)^n$. Intuitively this is confusing, but it says that the solid $S_n(1)$ is an increasingly small portion of the unit n-cube $[0,1]^n$. Pretty weird.

12.6.2. Compute the area of the region cut from the plane $x + y + z = a$ by the cylinder $x^2 + y^2 = a^2$.

Solution. Name the region S. Since the z values of the points of S lie on the plane $z = a - x - y$ and the projection of the region down to the xy-plane is a circle of radius a, one possible parametrization is

$$\Phi(r,\theta) = (r\cos(\theta), r\sin(\theta), a - r\cos(\theta) - r\sin(\theta)).$$

The magnitude of the normal can be computed as $||n|| = r\sqrt{3}$. Therefore the area of S is

$$\iint_S 1\,dS = \int_0^{2\pi}\int_0^a r\sqrt{3}\,dr\,d\theta = \pi\sqrt{3}a^2.$$

12.6.3. Compute the surface area of the portion of the sphere $x^2+y^2+z^2 = a^2$ lying within the cylinder $x^2 + y^2 = ay$, where $a > 0$.

Solution. Similarly, the x, y values of the parametrization will be determined by the circle the cylinder forms in the xy-plane. This circle is a circle of radius $a/2$ centered at $(0, a/2)$ since $x^2 + y^2 = ay$ can be rearranged to

$$x^2 + (y - a/2)^2 = a^2/4$$

by completing the square. The z-value of the cylinder is determined by the sphere, so we have $z = \pm\sqrt{a^2 - x^2 - y^2}$. Taking the positive square root, we can multiply the area by 2 at the end. The parametrization is

$$\Phi(x,y) = (x, y, \sqrt{a^2 - x^2 - y^2})$$

so that

$$||n|| = \sqrt{\frac{x^2}{a^2 - x^2 - y^2} + \frac{y^2}{a^2 - x^2 - y^2} + 1} = \frac{a}{\sqrt{a^2 - x^2 - y^2}}.$$

The domain of the parametrization T is the circle above. Now compute the area integral. Note that the radius of a circle at an angle $\theta \in [0, \pi]$ is determined by the equation $x^2 + y^2 = ay$, which can resolve in polar coordinates as $r = a \sin \theta$.

$$\begin{aligned}
\iint_S 1\, dS &= \iint_T \frac{2a}{\sqrt{a^2 - x^2 - y^2}}\, dx\, dy \\
&= \int_0^\pi \int_0^{a \sin \theta} \frac{2ar}{\sqrt{a^2 - r^2}}\, dr\, d\theta \\
&= \int_0^\pi 2a^2 - 2a\sqrt{a^2 - a^2 \sin^2(\theta)}\, d\theta \\
&= 2a^2 \int_0^\pi 1 - |\cos(\theta)|\, d\theta = 2a^2(\pi - 2)
\end{aligned}$$

12.6.5. A parametric surface S is described by

$$r(u, v) = (u \cos(v), u \sin(v), u^2)$$

where $0 \leq u \leq 4$ and $0 \leq v \leq 2\pi$. (a) Show that S is a portion of a surface of revolution. (b) Compute the fundamental vector product $\partial r/\partial u \times \partial r/\partial v$ in terms of u, v. (c) The area of S is $\pi(65\sqrt{65} - 1)/n$ where n is an integer. Compute n.

Solution. (a) This region is a surface of revolution since for each constant z value, the corresponding curve is a circle. Here the function we are revolving is $f(u) = u^2$.

(b) By the formula on page 429 in Apostol, the fundamental vector product is

$$\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} = (-2u^2 \cos(v), -2u^2 \sin(v), u).$$

(c) The area is computed by the integral

$$a(S) = 2\pi \int_0^4 u\sqrt{1 + 4u^2}\, du = \frac{2\pi}{12}(65\sqrt{65} - 1).$$

Therefore $n = 6$.

12.6.7. Compute the area of the portion of the conical surface $x^2 + y^2 = z^2$ which lies between the two planes $z = 0$ and $x + 2z = 3$.

Solution. The intersection of the cone with the plane $x+2z=3$ occurs for the following condition on x, y. Substituting $z=(3-x)/2$ into the cone equation, we obtain

$$x^2+y^2=\frac{(3-x)^2}{4}.$$

This is clearly a conic section and therefore will be some kind of ellipse. Rearrange the equation to obtain that the ellipse has the form

$$\frac{(x+1)^2}{4}+\frac{y^2}{3}=1.$$

The parametrization of the surface is

$$\Phi(x,y)=(x,y,\sqrt{x^2+y^2})$$

where x, y are in the ellipse above. Since we are graphing a function, the normal has the form

$$n=\left(\frac{x}{\sqrt{x^2+y^2}},\frac{y}{\sqrt{x^2+y^2}},-1\right)$$

with magnitude $||n||=\sqrt{2}$. Therefore $a(S)=\sqrt{2}a(E)$ where E the above ellipse. We conclude that $a(S)=2\pi\sqrt{6}$.

12.10.1. Let S denote the hemisphere $x^2+y^2+z^2=1$ with $z\geq 0$. Let $F(x,y,z)=(x,y,0)$. Let n be the unit outward normal of S. Compute $\iint_S F\cdot n\,dS$ using (a) a spherical parametrization and (b) a parametrization using the function $z=\sqrt{1-x^2-y^2}$.

Solution. (a)

12.10.5. Given a surface S which is a graph of a function $f(x,y)$ over a domain T, prove a simpler formula for the integral $\iint F\cdot dS$.

Solution. Given a parametrization of a surface $\Phi(x,y)=(x,y,f(x,y)$ where $(x,y)\in T$, then the normal vector is given by

$$n=\left(1,0,\frac{\partial f}{\partial x}\right)\times\left(0,1,\frac{\partial f}{\partial y}\right)=\left(-\frac{\partial f}{\partial x},-\frac{\partial f}{\partial y},1\right).$$

If $F=(P,Q,R)$, then by definition

$$\iint_S F\cdot n\,dS=\iint_T -P\frac{\partial f}{\partial x}-Q\frac{\partial f}{\partial y}+R\,dx\,dy.$$

12.10.6. Prove some more identities in the situation of 12.10.5.

Solution. (a) Given the normal n in 12.10.5, its magnitude is

$$||n|| = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}$$

so that for a scalar function φ, by definition

$$\iint_S \varphi(x, y, z)\, dS = \iint_T \varphi(x, y, f(x, y))\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dx\, dy.$$

(b) Note that in the problem $x = x$, $y = y$, and $z = f(x, y)$. By definition in Apostol (actually by definition of the wedge product), we have that

$$dy \wedge dz = \frac{\partial(y, z)}{\partial(x, y)}\, dx\, dy = \det \begin{bmatrix} 0 & \frac{\partial f}{\partial x} \\ 1 & \frac{\partial f}{\partial y} \end{bmatrix}\, dx\, dy = -\frac{\partial f}{\partial x}\, dx\, dy.$$

Therefore

$$\iint_S \varphi\, dy \wedge dz = -\iint_T \varphi(x, y, f(x, y))\frac{\partial f}{\partial x}\, dx\, dy.$$

(c) Similarly to the previous exercise,

$$dz \wedge dx = -\frac{\partial f}{\partial y}\, dx\, dy$$

so that by definition

$$\iint_S \varphi\, dz \wedge dx = -\iint_T \varphi(x, y, f(x, y))\, dx\, dy.$$

12.10.7. If S is the surface of a sphere of radius a centered at the origin, compute the value of

$$\iint_S xz\, dy \wedge dz + yz\, dz \wedge dx + x^2\, dx \wedge dy.$$

Use the outward normal.

Solution. Using the parametrization

$$\Phi(\theta, \varphi) = a(\cos(\theta)\sin(\varphi), \sin(\theta)\sin(\varphi), \cos(\varphi))$$

we see that the normal vector is

$$n = a^2 \sin(\varphi)(\cos(\theta)\sin(\varphi), \sin(\theta)\sin(\varphi), \cos(\varphi)).$$

Therefore the integral becomes

$$\begin{aligned}
&\iint_S xz\,dy \wedge dz + yz\,dz \wedge dx + x^2\,dx \wedge dy \\
&\qquad = a^4 \int_0^{\pi} \int_0^{2\pi} \sin(\varphi)(\cos^2(\theta)\sin^2(\varphi)\cos(\varphi) + \sin^2(\theta)\sin^2(\varphi)\cos(\varphi) \\
&\qquad\qquad + \cos^2(\theta)\sin^2(\varphi)\cos(\varphi))\,d\theta\,d\varphi \\
&\qquad = 3a^4\pi \int_0^{2\pi} \sin(\varphi)^3\cos(\varphi)\,d\varphi = 0.
\end{aligned}$$