Homework 13 Solutions
February 1, 2020

12.13.1. Transform the surface integral of $\nabla\times F$ where $F(x,y,z) = (y^2, xy, xz)$ over the hemisphere of $x^2+y^2+z^2 = 1$ with $z \geq 0$ into a line integral, and then evaluate the integral.

Solution. Call the surface S. The boundary of S is a circle of radius 1 in the xy-plane centered at the origin. By Stoke's theorem,

$$\iint_S \nabla\times F\cdot dS = \int_{\partial S} F\,ds = \int_0^{2\pi} (\sin^2(t), \cos(t)\sin(t), 0)\cdot(-\sin(t), \cos(t), 0)\,dt.$$

The integral evaluates to 0.

12.13.4. Compute the integral of $\nabla\times F$ where $F = (xz, -y, x^2y)$ over the surface S which sonsts of three faces not in the xz-plane of the tetrahedron bounded by the three coordinate planes, and the planes $3x+y+3z = 6$. The normal is outward.

Solution. The boundary of S is the intersection of the plane with the xz-plane, which is when $y = 0$. Thus it is the triangle formed by $3x+3z = 6$, or $z = 2-x$, $x = 0$, and $z = 0$. The integral can be computed using Stoke's theorem

$$\begin{aligned}
\iint_S \nabla\times F\cdot dS &= \int_{\partial S} F\cdot ds \\
&= \int_0^2 (0,0,0)\cdot(0,0,1)\,dt + \int_0^2 (0,0,0)\cdot(-1,0,0)\,dt \\
&+ \int_0^2 (t(2-t), 0, 0)\cdot(1,0,-1)\,dt \\
&= 4/3
\end{aligned}$$

12.13.5. Use Stoke's theorem to show that $\int_C (y,z,x)\cdot ds = \pi a^2\sqrt{3}$ where C is the curve of intersection between $x^2+y^2+z^2 = a^2$ and $x+y+z = 0$.

Solution. First, note that $\nabla\times F = (-1,-1,-1)$. Then to apply Stoke's theorem, we can choose any surface S with C as a boundary and evaluate the integral of $(-1,-1,-1)$ over S. Let S be the portion of the plane $x+y+z = 0$ inside the curve. Let T be the projection of S onto the xy-plane. Then a parametrization of S is $\Phi(x,y) = (x, y, -x-y)$ with domain T. The normal is therefore also $n = (-1,-1,-1)$ by the parametrization of a graph formula.

To find T substitute $z = -x - y$ into the equation for the sphere, to obtain $x^2 + y^2 + xy = a^2/2$. The relevant integral is

$$\iint_S (-1,-1,-1) \cdot dS = \iint_T (-1,-1,-1) \cdot \frac{1}{\sqrt{3}}(-1,-1,-1)\, dx\, dy = \sqrt{3}a(S)$$

Since S divides the sphere in half, then it is the same area as $x^2 + y^2 = a^2$ by a rotation. The area of this circle is πa^2, so that the integral is $\pi a^2 \sqrt{3}$.

12.13.6. Show that $\int_C (y+z)\, dx + (z+x)\, dy + (x+y)\, dz = 0$ where C is the curve of intersection of the cylinder $x^2 + y^2 = 2y$ and the plane $y = z$.

Solution. Using the same method as last exercise, we find a surface is C as a boundary and integrate $\nabla \times F$ over that surface S. We pick S to be the portion of the plane $z = y$ inside of $x^2 + y^2 = 2y$. The normal is $n = (0, -1, 1)$ and $\nabla \times F = (0,0,0)$, so therefore $\iint_S 0 \cdot dS = 0$ as desired.

12.13.11. If $r = (x, y, z)$ and $(P, Q, R) = a \times r$ where a is a constant, show that

$$\int_C (P, Q, R)\, ds = 2 \iint_S a \cdot n\, dS$$

where C is a curve bounding a surface S and n is a suitable normal.

Solution. By Stoke's theorem,

$$\int_C (P, Q, R)\, ds = \iint_S \nabla \times (P, Q, R) \cdot n\, dS.$$

Therefore it suffices to calculate the curl of (P, Q, R). Let $a = (a, b, c)$, at the risk of bad notation.

$$\begin{aligned} \nabla \times (P, Q, R) &= \nabla \times (a \times r) \\ &= \nabla \times (bz - cy, cx - az, ay - bx) \\ &= (a + a, b + b, c + c) = 2a \end{aligned}$$

Then the result follows.

12.13.13. (a) Use the formula for differentiating a product to show that

$$\frac{\partial}{\partial u}\left(p \frac{\partial X}{\partial v}\right) - \frac{\partial}{\partial v}\left(p \frac{\partial X}{\partial u}\right) = \frac{\partial p}{\partial u}\frac{\partial X}{\partial v} - \frac{\partial p}{\partial v}\frac{\partial X}{\partial u}.$$

(b) Now let $p(u, v) = P(X, Y, Z)$ where X, Y, Z are functions of u, v. Compute $\frac{\partial p}{\partial u}$ and $\frac{\partial p}{\partial v}$ by the chain rule and use part (a) to deduce a relation.

Solution. (a) By the product rule and equality of mixed partials in this case:

$$\begin{aligned}\frac{\partial}{\partial u}\left(p\frac{\partial X}{\partial v}\right)-\frac{\partial}{\partial v}\left(p\frac{\partial X}{\partial u}\right)&=p\frac{\partial^2 X}{\partial u\partial v}+\frac{\partial p}{\partial u}\frac{\partial X}{\partial v}-p\frac{\partial^2 X}{\partial v\partial u}-\frac{\partial p}{\partial v}\frac{\partial X}{\partial u}\\&=\frac{\partial p}{\partial u}\frac{\partial X}{\partial v}-\frac{\partial p}{\partial v}\frac{\partial X}{\partial u}.\end{aligned}$$

(b) The chain rule implies that

$$\begin{bmatrix}\frac{\partial p}{\partial u} & \frac{\partial p}{\partial v}\end{bmatrix}=\begin{bmatrix}\frac{\partial P}{\partial X} & \frac{\partial P}{\partial Y} & \frac{\partial P}{\partial Z}\end{bmatrix}\begin{bmatrix}\frac{\partial X}{\partial u} & \frac{\partial X}{\partial v}\\ \frac{\partial Y}{\partial u} & \frac{\partial Y}{\partial v}\\ \frac{\partial Z}{\partial u} & \frac{\partial Z}{\partial v}\end{bmatrix}.$$

Doing out the matrix multiplication and plugging into the right hand side of part (a), we obtain:

$$\begin{aligned}\frac{\partial p}{\partial u}\frac{\partial X}{\partial v}-\frac{\partial p}{\partial v}\frac{\partial X}{\partial u}&=\left(\sum\frac{\partial P}{\partial X_i}\frac{\partial X_i}{\partial u}\right)\left(\frac{\partial X}{\partial v}\right)-\left(\sum\frac{\partial P}{\partial X_i}\frac{\partial X_i}{\partial v}\right)\left(\frac{\partial X}{\partial u}\right)\\&=\frac{\partial P}{\partial Y}\frac{\partial Y}{\partial u}\frac{\partial X}{\partial v}+\frac{\partial P}{\partial z}\frac{\partial Z}{\partial u}\frac{\partial X}{\partial v}-\frac{\partial P}{\partial Z}\frac{\partial Y}{\partial v}\frac{\partial X}{\partial u}-\frac{\partial P}{\partial Z}\frac{\partial Z}{\partial v}\frac{\partial X}{\partial u}\\&=-\frac{\partial P}{\partial Y}\frac{\partial(X,Y)}{\partial(u,v)}+\frac{\partial P}{\partial Z}\frac{\partial(X,Z)}{\partial(u,v)}.\end{aligned}$$

Combining with part (a), we obtain a proof of (12.29).

12.15.1ac. For each of the following vector fields, determine the Jacobian matrix and compute the curl and divergence. (a) $F=(x^2+yz,y^2+xz,z^2+xy)$ (c) $F=(z+\sin(y),-z+x\cos(y),0)$

Solution. (a) $DF=\begin{bmatrix}2x & z & y\\ z & 2y & x\\ y & x & 2z\end{bmatrix}$, $\nabla\times F=(0,0,0)$, $\nabla\cdot F=2(x+y+z)$.

(c) $DF=\begin{bmatrix}\cos(y) & 1 & 0\\ -x\sin(y) & -1 & \cos(y)\\ 0 & 0 & 0\end{bmatrix}$, $\nabla\times F=(1,1,0)$, $\nabla\cdot F=-x\sin(y)$.

12.15.2. If $R=(x,y,z)$ and $r=||R||$, compute $\nabla\times f(r)R$, where f is a differentiable function.

Solution. By the formula on page 446, we have

$$\begin{aligned}\nabla\times f(r)R&=f(r)(\nabla\times R)+\nabla f(r)\times R\\&=0+\frac{1}{r}\frac{\partial f}{\partial r}R\times R=0\end{aligned}$$

12.15.4. Again let $R = (x, y, z)$ and $r = ||R||$. Find all integers n such that $\nabla \cdot (r^n R) = 0$.

Solution. Begin by calculating $\nabla \cdot (r^n R)$.

$$\begin{aligned}\nabla \cdot (r^n R) &= \sum_i \frac{\partial}{\partial x_i}(x_i r^n) \\ &= \sum_i r^n + n x_i^2 r^{n-2} \\ &= 3r^n + n(x^2 + y^2 + z^2) r^{n-2} \\ &= (3 + n) r^n\end{aligned}$$

From here it is clear that only when $n = -3$ is expression equal to 0.

12.15.8. Prove the identity

$$\nabla \cdot (F \times G) = G \cdot (\nabla \times F) - F \cdot (\nabla \times G).$$

Solution. This identity is proved by writing out each side entrywise.

$$\begin{aligned}\nabla \cdot (F \times G) &= \nabla \cdot (F_2 G_3 - F_3 G_2, F_3 G_1 - F_1 G_3, F_1 G_2 - F_2 G_1) \\ &= \frac{\partial F_2}{\partial x} G_3 + F_2 \frac{\partial G_3}{\partial x} - \frac{\partial F_3}{\partial x} G_2 - F_3 \frac{\partial G_2}{\partial x} \\ &+ \frac{\partial F_3}{\partial y} G_1 + F_3 \frac{\partial G_1}{\partial y} - \frac{\partial F_1}{\partial y} G_3 - F_1 \frac{\partial G_3}{\partial y} \\ &+ \frac{\partial F_1}{\partial z} G_2 + F_1 \frac{\partial G_2}{\partial z} - \frac{\partial F_2}{\partial z} G_1 - F_2 \frac{\partial G_1}{\partial z}\end{aligned}$$

On the other hand:

$$\begin{aligned}G \cdot (\nabla \times F) - F \cdot (\nabla \times G) &= (G_1, G_2, G_3) \cdot \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \\ &- (F_1, F_2, F_3) \cdot \left(\frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z}, \frac{\partial G_1}{\partial z} - \frac{\partial G_3}{\partial x}, \frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} \right) \\ &= \frac{\partial F_2}{\partial x} G_3 + F_2 \frac{\partial G_3}{\partial x} - \frac{\partial F_3}{\partial x} G_2 - F_3 \frac{\partial G_2}{\partial x} \\ &+ \frac{\partial F_3}{\partial y} G_1 + F_3 \frac{\partial G_1}{\partial y} - \frac{\partial F_1}{\partial y} G_3 - F_1 \frac{\partial G_3}{\partial y} \\ &+ \frac{\partial F_1}{\partial z} G_2 + F_1 \frac{\partial G_2}{\partial z} - \frac{\partial F_2}{\partial z} G_1 - F_2 \frac{\partial G_1}{\partial z}\end{aligned}$$

12.15.11. Let $V(x, y) = (y^c, x^c)$ where $c > 0$. Let $r(x, y) = (x, y)$. Let R be the plane region bounded by a piecewise smooth Jordan curve C. Compute $\nabla \cdot (V \times r)$, and $\nabla \times (V \times r)$. Use Green's theorem to show that $\int_C V \times r \cdot ds = 0$.

Solution. First we compute $V \times r$, where we view them in $\mathbb{R}^3$ with z-component 0. Then

$$V \times r = (0, 0, y^{c+1} - x^{c+1}).$$

Then $\nabla \cdot (V \times r) = 0 + 0 + 0 = 0$ and $\nabla \times (V \times r) = (c+1)(y^c, x^c, 0) = (c+1)V$. We apply Green's theorem (well really Stoke's theorem), so that $\int_C V \times r \, dS = \iint_S \nabla \times (V \times r) \, dS$. The latter integral is computed as follows, since S is a flat surface with unit normal $n = (0, 0, 1)$.

$$\iint_S (c+1)(V) \cdot (0, 0, 1) \, dS = \iint_S 0 \, dS = 0.$$

12.21.1. Let S be the surface of then unit cube $\partial[0, 1]^3$. Let n be the unit outer normal of S. If $F = (x^2, y^2, z^2)$, use the divergence theorem to evaluate the surface integral $\iint F \cdot n \, dS$. Verify the result by computing the integral directly.

Solution. By the divergence theorem,

$$\iint_S F \cdot n \, dS = \iiint_{[0,1]^3} 2(x + y + z) \, dV = 3.$$

Manually, the integrals on the three sides on the coordinate planes are 0 since the normal is perpendicular to the parametrization for the side. On the three other sides, $F \cdot n = 1$ so that you add up the areas of the 3 sides and you get 3.

12.21.2. The sphere of radius 5 centered at the origin is intersected by the plane $z = 3$. The smaller portion forms a solid V closed by a surface S_0, made up of the sphere part S_1 and the plane part S_2. Compute

$$\iint_S (xz, yz, 1) \, dS$$

for (a) $S = S_1$, (b) $S = S_2$, and (c) $S = S_0$. Solve for part (c) using the parts of (a) and (b), and also by the divergence theorem.

Solution. (a) Note that the surface S_1 is the graph of $z = \sqrt{25 - x^2 - y^2}$ over the circle $x^2 + y^2 \leq 16$ in the plane. Therefore

$$\begin{aligned}
\iint_{S_1} (xz, yz, 1) \cdot \, dS &= \iint_T (xz, yz, 1) \cdot (x/z, y/z, 1) \, dx \, dy \\
&= \iint_T x^2 + y^2 + 1 \, dx \, dy \\
&= 2\pi \int_0^4 (r^2 + 1) r \, dr = 144\pi
\end{aligned}$$

(b) Similarly, the planar region is the graph of $z = 3$ over the region $x^2+y^2 \leq 16$ in the plane, so that with upward normal, the integral can be computed.

$$\iint_{S_2} (xz, yz, 1)\, dS = \iint_T (xz, yz, 1) \cdot (0, 0, 1)\, dx\, dy = \iint_T 1\, dx\, dy = 16\pi$$

(c) On the one hand, the outward normal on S_0 means that

$$\iint_{S_0} F \cdot dS = \iint_{S_1} F \cdot dS - \iint_{S_2} F \cdot dS = 144\pi - 16\pi = 128\pi.$$

Let W be the interior of S_0. Then on the other hand, the divergence theorem implies that

$$\begin{aligned}
\iint_{S_0} F \cdot dS &= \iiint_W 2z\, dz\, dx\, dy \\
&= \iint_T \int_3^{\sqrt{25-x^2-y^2}} 2z\, dz\, dx\, dy \\
&= \iint_T 16 - x^2 - y^2\, dx\, dy \\
&= 2\pi \int_0^4 r(16 - r^2)\, dr = 128\pi
\end{aligned}$$

12.21.4,6. Let $\frac{\partial f}{\partial n} = \nabla f \cdot n$ and assume a region V in $\mathbb{R}^3$ has boundary S which is a closed surface. Then prove the following identities. (4) $\iint_S \frac{\partial f}{\partial n}\, dS = \iiint_V \nabla^2 f\, dx\, dy\, dz$ and (6) $\iint_S f \frac{\partial g}{\partial n}\, dS = \iiint_V f\nabla^2 g + \nabla f \cdot \nabla g\, dx\, dy\, dz$

Solution. (4) By the divergence theorem:

$$\iint_S \frac{\partial f}{\partial n}\, dS = \iint_S \nabla f \cdot dS = \iiint_V \nabla \cdot \nabla f\, dx\, dy\, dz = \iiint_V \nabla^2 f\, dx\, dy\, dz.$$

(6) Again by the divergence theorem and the divergence of a product formula:

$$\begin{aligned}
\iint_S f \frac{\partial g}{\partial n}\, dS &= \iint_s f\nabla g \cdot dS \\
&= \iiint_V \nabla \cdot (f\nabla g)\, dx\, dy\, dz \\
&= \iiint_V f\nabla^2 g + \nabla f \cdot \nabla g\, dx\, dy\, dz
\end{aligned}$$

These two summands at the end can be integrated separately if desired.

12.21.11. Let V be a convex region in $\mathbb{R}^3$ whose boundary is a closed surface S and let n be the unit outer normal of S. Let F and G be two continuously differentiable vector fields such that $\nabla \times F = \nabla \times G$ and $\nabla \cdot F = \nabla \cdot G$ everywhere on V and such that $F \cdot n = G \cdot n$ on S. Prove that $F = G$ everywhere on V.

Solution. Let $H = F - G$. Since all the above relations are linear, we have that $\nabla \times H = 0$, $\nabla \cdot H = 0$, and $H \cdot n = 0$. It suffices to show that $H = 0$. Since V is convex, then $\nabla \times H = 0$ implies that H is conservative, so that $H = \nabla f$. Now, note that

$$||H||^2 = \nabla f \cdot \nabla f = \nabla \cdot f \nabla f - f \nabla \cdot \nabla f = \nabla \cdot f \nabla f$$

since $\nabla \cdot H = 0$. Now by the divergence theorem

$$\iiint_V ||\nabla f||^2 \, dx \, dy \, dz = \iiint_V \nabla \cdot f \nabla f \, dx \, dy \, dz = \iint_S f \nabla f \cdot dS$$

But $H \cdot n = 0$ then $f \nabla f \cdot n = 0$ as well. Therefore this integral is zero, and we conclude that

$$\iiint_V ||H||^2 \, dx \, dy \, dz = 0.$$

Since $||H||^2$ is a nonnegative function on V, then $H = 0$ as desired.

6.3.1. Solve the differential equation $y' - 3y = e^{2x}$ on all of $\mathbb{R}$ when $y(0) = 0$.

Solution. The integrating factor is $A(x) = \int_0^x -3 \, dt = (-3t)_0^x = -3x$. Therefore by theorem 6.1, the unique solution on $\mathbb{R}$ to the IVP is

$$y = e^{3x} \int_0^x e^{2t} e^{-3t} \, dt = e^{3x} \int_0^x e^{-t} \, dt = e^{3x}(-e^{-x} + 1) = e^{3x} - e^{2x}.$$

6.3.5. A curve with equation $y = f(x)$ passes through the origin. Lines drawn parallel to the coordinate axes through an arbitrary point of the curve form a rectangle with two sides on the axes. The curve divides every such rectangle into two regions A and B, one of which has an area equal to n times the other. Find the function f.

Solution. Consider the point $(x, f(x))$ on the curve. The area below the curve is determined by the integral $\int_0^x f(t) \, dt$ while the area above the integral is determined by $\int_0^x f(x) - f(t) \, dt$. The relation between them is

$$n \int_0^x f(t) \, dt = \int_0^x f(x) - f(t) \, dt.$$

The fundamental theorem of calculus transforms this equation into

$$nf(x) = f(x) + xf'(x) - f(x).$$

Therefore it suffices to solve the separable differential equation

$$\frac{dy}{dx} = \frac{n}{x}y.$$

The usual method yields $\ln(y) = n\ln(x) + c$ so that $y = cx^n$.

6.3.7,8,9,10. Find all solutions of the following differential equations on $\mathbb{R}$. (7) $y'' - 4y = 0$ (8) $y'' + 4y = 0$, (9) $y'' - 2y + 5y = 0$ (10) $y'' + 2y' + y = 0$.

Solution. These solutions use facts in theorem 6.2 and the ensuing discussion below the theorem.

(7) The roots of the characteristic equation $r^2 - 4 = 0$ are $r = \pm 2$ so that the solution space of the equation is spanned by e^{2x} and e^{-2x}. Therefore a general solution has the form $y = c_1 e^{2x} + c_2 e^{-2x}$.

(8) The roots of the characteristic equation $r^2 + 4 = 0$ are $r = \pm 2i$. In this case the discriminant is negative, so the solution space of the equation is spanned by $\cos(2x)$ and $\sin(2x)$ over $\mathbb{R}$. Therefore a general solution has the form $y = c_1\cos(2x) + c_2\sin(2x)$.

(9) The roots of the characteristic equation $r^2 - 2r + 5 = 0$ are $r = 1 \pm 2i$. Again the discriminant is negative, so the solution space is spanned by $e^x\cos(2x)$ and $e^x\sin(2x)$ over the reals. Therefore a general solution has the form $y = c_1 e^x\cos(2x) + c_2 e^x\sin(2x)$.

(10) The roots of the characteristic equation $r^2 + 2r + 1 = 0$ are $r = -1$ with multiplicity 2. The discriminant in this case vanishes, so that the solution space is spanned by e^{-x} and xe^{-x}. Therefore the general solution has the form $y = c_1 e^{-x} + c_2 xe^{-x}$.

7.4.2. Verify each of the following differentiation rules for matrix functions, assuming P and Q are differentiable. (a) $(P + Q)' = P' + Q'$ (b) $(PQ)' = PQ' + P'Q$ (c) $(Q^{-1})' = -Q^{-1}Q'Q^{-1}$ (d) $(PQ^{-1})' = -PQ^{-1}Q'Q^{-1} + P'Q^{-1}$

Solution. (a) This relation follows readily from the fact that the derivative is linear on each entry. On the ijth entry, we have $(p_{ij} + q_{ij})' = p'_{ij} + q'_{ij}$. Since $(PQ)'$ and $P' + Q'$ have equal entries, they are equal matrices.

(b) Similarly, we can verify this relation on the entries.

$$\left(\sum_k p_{ik}qkj\right)' = \sum_k p'_{ik}q_{kj} + p_{ik}q'_{kj} = \sum_k p'_{ik}q_{kj} + \sum_k p_{ik}q'_{kj}.$$

Notice the the left hand side is the ijth entry of $(PQ)'$ and the right hand side is the ijth entry of $P'Q + PQ'$.

(c) Since Q^{-1} exists, we know that $QQ^{-1} = I$. Using the product rule on this equation, we obtain $Q'Q^{-1} + Q(Q^{-1})' = 0$. Solving for $(Q^{-1})'$, the desired relation

$$(Q^{-1})' = -Q^{-1}Q'Q^{-1}$$

is obtained.

(d) This is a direct combination of (b) and (c).

7.4.3. (a) Prove formulas for $(P^2)'$ and $(P^3)'$. (b) Guess the formula for $(P^k)'$ and prove it by induction.

Solution. (a) By the product rule $(P^2)' = P'P + PP'$. For the next power:

$$(P^3)' = (P^2)'P + P^2P' = P'P^2 + PP'P + P^2P'.$$

(b) We claim that $(P^k)' = \sum_{i=1}^{k} P^{i-1}P'P^{k-i}$. The base case is the previous part of the problem. Assume the case for $k-1$. Then

$$\begin{aligned}(P^k)' = (P^{k-1})'P + P^{k-1}P' &= \left(\sum_{i=1}^{k-1} P^{i-1}P'P^{k-i-1}\right) P + P^{k-1}P' \\ = \sum_{i=1}^{k-1} P^{i-1}P'P^{k-i} + P^{k-1}P' &= \sum_{i=1}^{k} P^{i-1}P'P^{k-i}\end{aligned}$$

7.4.8. Prove that $||A + B|| \leq ||A|| + ||B||$ and $|c| \cdot ||A|| = ||cA||$.

Solution. The triangle inequality for this matrix norm follows from the triangle inequality for real numbers.

$$\begin{aligned}||A + B|| &= \sum_{i,j} |a_{ij} + b_{ij}| \leq \sum_{ij} |a_{ij}| + |b_{ij}| \\ &= \sum_{i,j} |a_{ij}| + \sum_{i,j} |b_{ij}| = ||A|| + ||B||\end{aligned}$$

For the second identity:

$$||cA|| = \sum_{i,j} |ca_{ij}| = |c| \sum_{i,j} |a_{ij}| = |c| \cdot ||A||.$$

7.4.9. If a matrix function P is integrable on an interval $[a, b]$, prove that $\left|\int_a^b P(t)\, dt\right| \leq \int_a^b |P(t)|\, dt$.

Solution. This inequality also follows from the case for real integrable functions.

$$\begin{aligned}\left|\int_a^b P(t)\, dt\right| &= \sum_{i,j} \left|\int_a^b p_{ij}(t)\, dt\right| \leq \sum_{i,j} \int_a^b |p_{ij}(t)|\, dt \\ &= \int_a^b \sum_{i,j} |p_{ij}(t)|\, dt = \int_a^b |P(t)|\, dt\end{aligned}$$