Homework 14 Solutions
February 1, 2020

7.4.10. Let D be an $n \times n$ diagonal matrix with entries $d_{ii} = \lambda_i$. Prove that the matrix series $\sum_{k=0}^{\infty} D^k/k!$ converges and is also a diagonal matrix.

Solution. We claim that $\sum_{k=0}^{\infty} D^k/k!$ is the diagonal matrix with entries e^{λ_i}, which we call A. Indeed D^k is diagonal with entries λ_i^k. Therefore since $e^x = \sum_{n=0}^{\infty} x^n/n!$, we pick high enough N such that for each i,

$$\left| \sum_{k=0}^{N} \frac{\lambda_i^k}{k!} - e^{\lambda_i} \right| < \varepsilon/n.$$

Then

$$\left| \sum_{k=0}^{N} \frac{D^k}{k!} - A \right| \leq \sum_i \left| \sum_{k=0}^{N} \frac{\lambda_i^k}{k!} - e^{\lambda_i} \right| < \varepsilon.$$

This proves convergence of the matrix series.

7.4.12. Assume that the matrix series $\sum_{k=1}^{\infty} C_k$ converges, where each C_k is an $n \times n$ matrix. Prove that the matrix series $\sum_{k=1}^{\infty} AC_kB$ converges to $A\left(\sum_{k=1}^{\infty} C_k\right)B$.

Solution. Let S_N denote the Nth partial sum matrix and let S denote the infinite series. Pick N high enough so that $|S_N - S| < \varepsilon/(|A||B|)$. Then

$$\begin{aligned} \left| \sum_{k=1}^{N} (AC_kB) - ASB \right| &= |AS_NB - ASB| \\ &= |A(S_N - S)B| \\ &\leq |A||S_N - S||B| \\ &< |A||B| \frac{\varepsilon}{|A||B|} = \varepsilon. \end{aligned}$$

7.12.1,2,4. (a) Express the powers of the following matrices as a linear combination of I and A. (b) Calculate e^{tA}. (1) $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ (2) $A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$ (4) $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

Solution. (1a) By the Cayley-Hamilton theorem, $A^2 = 2A - I$, since A satisfies its own characteristic polynomial $\lambda^2 - 2\lambda + 1$. Now we claim that $A^n =$

$nA - (n-1)I$. By induction, assume the formula for A^{n-1}. Then

$$\begin{aligned} A^n &= A((n-1)A - (n-2)I) = (n-1)(2A - I) - (n-2)A \\ &= (2(n-1) - (n-2))A - (n-1)I = nA - (n-1)I \end{aligned}$$

as desired.

(1b) Note that A is not diagonalizable, but note that $N = A - I = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ is nilpotent. Therefore

$$e^{tA} = e^{t(I+N)} = e^{tI}e^{tN} = e^t(I + tN) = \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix}$$

(2a) Again the Cayley-Hamilton theorem $A^2 = 2A - 2I$. We claim that $A^n = (2^n - 1)A - (2^n - 2)I$ Again by induction,

$$\begin{aligned} A^n &= (2^{n-1} - 1)A^2 - (2^{n-1} - 2)A = (2^{n-1} - 1)(3A - 2I) - (2^{n-1} - 2)A \\ &= (3(2^{n-1}) - 2^{n-1} - 3 + 2)A - (2^n - 2)I = (2^n - 1)A - (2^n - 2)I. \end{aligned}$$

(2b) Diagonalize A by

$$A = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Now we can take the exponent of the corresponding diagonal matrix and conjugate by the change of basis matrix.

$$e^{tA} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} e^t & 0 \\ -e^t + e^{2t} & e^{2t} \end{bmatrix}$$

(4a) It is clear that $A^2 = I$, so that $A^n = I$ if n is even and $A^n = A$ if n is odd.

(4b) Since A is diagonal, then $e^{tA} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^t \end{bmatrix}$.

7.12.7. Compute the derivative of $e^{A(t)}$ where $A(t) = \begin{bmatrix} 1 & t \\ 0 & 0 \end{bmatrix}$, and show that it is neither $A'(t)e^{A(t)}$ nor $e^{A(t)}A'(t)$.

Solution. First, diagonalize the matrix function. The usual method shows that

$$A(t) = \begin{bmatrix} -t & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & t \end{bmatrix}.$$

Therefore

$$e^{A(t)} = \begin{bmatrix} -t & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & t \end{bmatrix} = \begin{bmatrix} e & (e-1)t \\ 0 & 1 \end{bmatrix}.$$

Taking the derivative, $(e^{A(t)})' = \begin{bmatrix} 0 & e-1 \\ 0 & 0 \end{bmatrix}$.

We can see directly that this is not equal to $A'(t)e^{A(t)}$ or $e^{A(t)}A'(t)$. Calculating each,

$$A'(t)e^{A(t)} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e & et \\ 0 & 0 \end{bmatrix} = 0$$

and

$$e^{A(t)}A'(t) = \begin{bmatrix} e & et \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & e \\ 0 & 0 \end{bmatrix}.$$

None of these are equal.

7.12.8. Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. (a) Calculate A^n and express A^3 in terms of I, A, and A^2. (b) Calculate e^{tA}.

Solution. (a) We claim that this matrix is nilpotent. Indeed it easy to check $A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and that $A^3 = 0$. Therefore $A^n = 0$ for $n \geq 3$. Trivially, then $A^3 = 0A^2 + 0A + 0I$.

(b) Since A is nilpotent, then $e^{tA} = I + tA + t^2A^2/2 = \begin{bmatrix} 1 & t & t + t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$.

7.12.13. Compute e^Ae^B, e^Be^A, and e^{A+B} when $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$. Note that these three matrices are distinct.

Solution. First, e^A is simply $e^{A(1)}$ where $A(t)$ is defined in exercise 7.12.7. There we showed that $e^A = \begin{bmatrix} e & e-1 \\ 0 & 1 \end{bmatrix}$. Similarly $e^B = e^{A(-1)} = \begin{bmatrix} e & 1-e \\ 0 & 1 \end{bmatrix}$. Then $e^Ae^B = \begin{bmatrix} e^2 & -e^2+2e-1 \\ 0 & 1 \end{bmatrix}$ and $e^Be^A = \begin{bmatrix} e & e^2-2e+1 \\ 0 & 1 \end{bmatrix}$. Finally, $A + B = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$, so that $e^{A+B} = \begin{bmatrix} e^2 & 0 \\ 0 & 1 \end{bmatrix}$. All of these matrices are distinct.

Extra Problem # 1. Let A be an $n \times n$ matrix which leaves a subspace $E \subseteq \mathbb{R}^n$ invariant. Prove that e^A also leaves E invariant.

Solution. Let $S_n(A)$ be the partial sums of e^A, so that $|S_n - e^A| \to 0$. First, note that $S_n(A)$ leaves E invariant. Let $x \in E$. Then

$$S_n(A)x = \sum_{k=0}^{n} \frac{A^k}{k!} x$$

which is a linear combination of elements of E. Since E is a subspace, then $S_n(A)x \in E$.

Now we claim that $S_n(A)x \to e^A x$ as vectors in $\mathbb{R}^n$. In fact by submultiplicativity,

$$|S_n(A)x - e^A x| \leq |S_n - e^A||x| \to 0.$$

Finally, it suffices to show that the limit of vectors in E is also in E. This follows from the fact that E is closed and closed sets contain limit points. It is clear that E is a closed set (use the general version the distance from a point to a plane formula). Suppose for contradiction that $e^A x \notin E$. Since E^c is open, let $B_\delta(e^A x)$ be a ball contained in E^c centered at $e^A x$. Since $S_n(A)x \to e^A x$, there exists an N such that for all $n > N$, $|S_n(A)x - e^A x| < \delta$. This would imply that $S_n(A)x \in B_\delta(e^A x) \subseteq E^c$ for some n, which is a contradiction.

Extra Problem # 2. With the same assumptions as above, prove that if $x(t)$ is a solution of $x'(t) = Ax(t)$, $x(0) = x_0$ with $x_0 \in E$, then we can conclude that $x(t) \in E$, for all $t \in \mathbb{R}$.

Solution. By the main theorem, we know that the unique solution to this vector differential equation is $x(t) = e^{At}x_0$. Since A leaves E invariant, then so does tA, and therefore so does e^{At}. Thus $e^{tA}x_0 \in E$ since $x_0 \in E$. This shows that $x(t) \in E$ for all t.

Extra Problem # 3. Let $\phi(t, x_0)$ denote the solution of the initial value problem above. Here $x_0 \in \mathbb{R}^n$ however. Prove that the solution is continuous with respect to the initial conditions, in that for each fixed t, we have the following $\lim_{y \to x_0} \phi(t, y) = \phi(t, x_0)$.

Solution. The claim follows from the main theorem, that $\phi(t, x_0) = e^{tA}x_0$. For a fixed t, we show that $\lim_{y \to x_0} e^{tA}y = e^{tA}x_0$, or in other words, given any $\varepsilon > 0$, we can find δ such that $|y - x_0| < \delta$ implies $|e^{At}y - e^{At}x_0| < \varepsilon$. Since t is fixed and $y \to x_0$, pick $\delta = \varepsilon / |e^{At}|$. Then

$$|e^{At}y - e^{At}x_0| \leq |e^{At}||y - x_0| < |e^{At}| \frac{\varepsilon}{|e^{At}|} = \varepsilon.$$

This proves the claim.