Homework 1 Solutions
February 1, 2020

4.4.1.

(a) If T has an eigenvalue λ, prove that aT has the eigenvalue $a\lambda$.

(b) If x is an eigenvector for both T_1 and T_2, prove that x is also an eigenvector for $aT_1 + bT_2$. How are the eigenvalues related?

Solution. For (a), let x be an eigenvector of T corresponding to λ. We show that $(aT)(x) = (a\lambda)x$. Indeed

$$(aT)(x) = aT(x) = a\lambda x = (a\lambda)x.$$

This proves the claim.

For (b), let x be an eigenvector corresponding to λ_1 and λ_2 (for T_1 and T_2 respectively). We claim that x is an eigenvector corresponding to $\lambda = a\lambda_1 + b\lambda_2$. Indeed

$$(aT_1 + bT_2)(x) = aT_1(x) + bT_2(x) = a\lambda_1 x + b\lambda_2 x = (a\lambda_1 + b\lambda_2)x.$$

4.4.4. If $T: V \to V$ has the property that T^2 has a nonnegative eigenvalue λ^2, prove that at least one of λ or $-\lambda$ is an eigenvalue for T.

Solution. By definition, the transformation $T^2 - \lambda^2 I$ has a nontrivial kernel. Let $x \in \ker(T^2 - \lambda^2 I)$. Then $T^2 - \lambda^2 I = (T - \lambda I)(T + \lambda I)$, not as multiplication but as function composition. Then either $x \in \ker(T + \lambda I)$ or $(T + \lambda I)(x) \in \ker(T - \lambda I)$. In the first case, then $-\lambda$ is an eigenvalue for x of T. In the second case, we can assume that $x \notin \ker(T + \lambda I)$. Then λ is an eigenvalue for T corresponding to the vector $(T + \lambda I)(x) \neq 0$.

4.4.5. Let V be the linear space of all real functions differentiable on $(0, 1)$. If $f \in V$, define $T(f)(t) = tf'(t)$ for all $t \in (0, 1)$. Prove that every real λ is an eigenvalue for T, and determine the eigenfunctions corresponding to λ.

Solution. An eigenfunction f for T corresponding to λ would satisfy the differential equation $tf'(t) = \lambda f(t)$. We can solve this by separation of variables (which I omit, but you should write in your solution), so that $f(t) = t^\lambda$. Indeed this linear function can be seen to satisfy the desired diff eq. Note that t^λ is well-defined on $(0, 1)$ for all λ.

4.4.7. Let V be the linear space of all functions continuous on $(-\infty, \infty)$ and such that the integral $\int_{-\infty}^{x} f(t)\,dt$ exists for all real x. If $f \in V$, let $T(f)(x) = \int_{-\infty}^{x} f(t)\,dt$. Prove that every positive λ is an eigenvalue for T and determine the eigenfunctions corresponding to λ.

Solution. An eigenfunction for T would satisfy the relation

$$\int_{-\infty}^{x} f(t)\,dt = \lambda f(t).$$

Taking the derivative, the fundamental theorem of calculus states that f needs to satisfy

$$f(x) = \lambda f'(x).$$

Solving this diff eq by separation of variables, we see that $f(x) = e^{x/\lambda}$ is an eigenfunction for T corresponding to $\lambda > 0$.

4.4.11. Assume that a linear transformation T has two eigenvectors x and y belonging to distinct eigenvalues λ and μ. If $ax + by$ is an eigenvector of T, prove that $a = 0$ or $b = 0$.

Solution. Let $ax + by$ correspond to the eigenvalue γ. Then on the one hand

$$T(ax + by) = \gamma(ax + by).$$

On the other hand,

$$T(ax + by) = a\lambda x + b\mu y.$$

Therefore $\gamma(ax + by) = a\lambda x + b\mu y$. Note that since $\lambda \neq \mu$, then $x \notin \text{span}(y)$, so that x and y are independent. The previous equation can be turned into the relation

$$a(\gamma - \lambda)x + b(\gamma - \mu)y = 0.$$

By the definition of independence, if $a, b \neq 0$, then $\gamma = \lambda$ and $\gamma = \mu$. This is a contradiction, so that either $a = 0$ or $b = 0$.