Homework 3 Solutions
February 1, 2020

5.11.1. Determine whether a bunch of matrices are symmetric, skew-symmetric, Hermitian, or skew-Hermitian.

Solution. a. symmetric b. skew-Hermitian c. Hermitian d. skew-symmetric

5.11.2. a. Verify that the 2×2 rotation matrix is an orthogonal matrix. b. Prove that it actually rotates vectors by θ.

Solution. a. This follows from $\sin^2 + \cos^2 = 1$. b. This follows from $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$ and $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$.

5.11.3. Show that a bunch of matrices do certain things the textbook is claiming.

Solution. a. $A(x, y, z) = (x, y, -z)$ which is the opposite side of the xy-plane.
b. $A(x, y, z) = (x, -y, -z)$ which is on the opposite side of the x axis.
c. $A(x, y, z) = (-x, -y, -z)$ which is on the opposite side of the origin.
d. This one keeps the same x but rotates y and z, which means you're rotating around the x axis. Also note that $(1, 0, 0)$ is an eigenvector with eigenvalue 1.
e. You multiply (b) and (d).

5.11.4. a. If a 2×2 matrix A is proper, show that it is a rotation matrix for some θ. b. Show that two matrices are not proper, and find all nonproper 2×2 matrices.

Solution. a. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an orthogonal matrix, then (a, b) and (c, d) are orthogonal unit vectors by the $AA^T = I$ criterion. Thus $(c, d) = \pm(-b, a)$. If $\det A = 1$, this restricts $(c, d) = (-b, a)$. Since (a, b) is a unit vector, then it is on the unit circle and has the form $(\cos(\theta), \sin(\theta))$. Therefore A is a rotation matrix.
b. The two matrices are improper since they are orthogonal with determinant -1. By the above argument, we can see all improper 2×2 matrices are of the form $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, namely a rotation matrix followed by a reflection.

5.11.6, 5.11.7. Diagonalize some matrices.

Solution. 6. $D = \begin{bmatrix} -2i & 0 \\ 0 & 2i \end{bmatrix}$, $C = \frac{1}{\sqrt{2}} \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}$
7. $D = \begin{bmatrix} -4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ $C = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$ where $v_1 = \frac{1}{\sqrt{35}}(-3, 5, 1)$, $v_2 = \frac{1}{\sqrt{10}}(1, 0, 3)$, and $v_3 = \frac{1}{\sqrt{14}}(-3, -2, 1)$.

5.11.13. If A is a real skew-symmetric matrix, prove that $I - A$ and $I + A$ are nonsingular and that $(I - A)(I + A)^{-1}$ is orthogonal.

Solution. Since A is real skew-symmetric, then it has only real eigenvalue $\lambda = 0$ if any. Then $\lambda = \pm 1$ are not eigenvalues, so by definition $A \pm I$ is nonsingular. Now we show that $B = (I - A)(I + A)^{-1}$ is orthogonal by showing $B^T = B^{-1}$.

This is true since

$$B^T = (I - A^T)(I + A^T)^{-1} = (I + A)(I - A)^{-1} = B^{-1}.$$

5.15.1, 5.15.5. For each of the following quadratic forms, find a symmetric matrix A for it, find the eigenvalues of A, find an orthonormal set of eigenvectors, and an orthogonal diagonalizing matrix.

1. $4x_1 + 4x_1x_2 + x_2^2$
5. $x_1^2 + x_1x_2 + x_1x_3 + x_2x_3$

Solution. 1. $A = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$ with eigenvalues $\lambda = 0, 5$ and orthogonal eigenbasis $\frac{1}{\sqrt{5}}(-1, 2)$ and $\frac{1}{\sqrt{5}}(2, 1)$.

5. $A = \frac{1}{2}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ with eigenvalues $\lambda = -1/2, 0, 3/2$ and eigenbasis $\frac{1}{\sqrt{2}}(0, -1, 1)$, $\frac{1}{\sqrt{3}}(-1, 1, 1)$, and $\frac{1}{\sqrt{6}}(2, 1, 1)$.

5.15.8, 5.15.9. Make a sketch of the following conic sections.

8. $y^2 - 2xy + 2x^2 - 5 = 0$
9. $y^2 - 2xy + 5x = 0$

Solution. You find the eigenbasis, change the variables, then sketch whether you get a rotated parabola, ellipse, or circle, hyperbola, or two lines. Did you know that a parabola is just an ellipse with one end at infinity?

8. rotated ellipse
9. rotated hyperbola

5.20.1a. Let $T : V \to V$ be the transformation given by $T(x) = cx$ for some fixed scalar c. Prove that T is unitary iff $|c| = 1$.

Solution. If T is unitary and has an eigenvalue λ then $|\lambda| = 1$ by theorem 5.16. For $T(x) = cx$, c is an eigenvalue trivially, so that $|c| = 1$.

5.20.1b. If V is one-dimensional, prove that only unitary transformation on V are those described in (a). If V is real, then $c = \pm 1$.

Solution. Since V is one-dimensional, then it has a basis element e, and therefore $T(e) = ce$ for some scalar c. Since e is a basis $T(x) = cx$ for all $x \in V$. Therefore (a) applies that if T is unitary then $|c| = 1$. If the underyling field is $\mathbb{R}$, then the only scalars with magnitude one are $c = \pm 1$.

5.20.2. Prove the following statements about a real orthogonal $n \times n$ matrix A. (a) If λ is a real eigenvalue of A, then $\lambda = \pm 1$. (b) If λ is a complex eigenvalue of A, then the complex conjugate $\overline{\lambda}$ is also an eigenvalue of A. (c) If n is odd, then A has at least one real eigenvalue.

Solution. (a) Again by theorem 5.16, then any eigenvalue of an orthogonal matrix has magnitude 1. The only scalars in $\mathbb{R}$ with that property are ± 1. (b) Since A is a real matrix and all complex roots of real polynomials come in conjugate pairs, this claim follows easily. (c) If n is odd, then the characteristic polynomial is off odd degree. All odd degree real polynomials have at least one real solution by the intermediate value theorem.

5.20.3. Let V be a real Euclidean space of dimension n. An orthogonal transformation $T : V \to V$ with determinent 1 is called a rotation. If n is odd, prove that 1 is an eigenvalue for T.

Solution. Since V is odd dimensional, then T has at least one real eigenvalue λ. Since T is orthogonal, then we know that $|\lambda| = 1$, so that $\lambda = \pm 1$. Since the complex eigenvalue come in conjugate pairs, then there are an odd number of real eigenvalues. Since $\det A = 1$, then -1 can only be a real eigenvalue an even number of times. Since even numbers are not odd numbers, there must be a left over real eigenvalue not equal to -1, so therefore $\lambda = 1$ is an eigenvalue.

1a.. For any square matrix, show that $A + A^T$ is symmetric and $A - A^T$ is skew symmetric.

Solution. Note that

$$(A + A^T)^T = A^T + A^{T^T} = A^T + A.$$

For the skew-symmetric case

$$(A - A^T)^T = A^T - A = -(A - A^T).$$

1b.. Show that any square matrix can be expressed uniquely as a sum of one symmetric and one skew-symmetric matrix.

Solution. We have at least one such decomposition by 1a since

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T).$$

To show that this decomposition is unique, assume $A = B + C = B' + C'$ where B, B' are symmetric and C, C' are skew-symmetric. Then $B + C = B' + C'$ and thus $(B - B') + (C - C') = 0$. Note that $B - B'$ and $C - C'$ are symmetric and skew-symmetric respectively. Thus if we can show the 0 matrix has a unique decomposition, then $B - B' = 0$ and $C - C' = 0$ and the proof will be done.

Now let $B + C = 0$ be such a decomposition of the 0 matrix. Taking the transpose, we obtain that $B - C = 0$ as well. Adding the equations yields $2B = 0$ so that $B = 0$ and therefore $C = 0$. Thus 0 has a unique decomposition, and by the above argument so does any general A.

1c.. Show that if n is odd, then any $n \times n$ skew-symmetric matrix has determinent 0.

Solution. Recall that any real eigenvalue of a skew-symmetric matrix must be 0. If n is odd, then the matrix A has at least one real eigenvalue. Since $\det A = \prod_i \lambda_i$, and at least one $\lambda_i = 0$, then $\det A = 0$.

1d.. Show that any nonzero 3×3 skew-symmetric matrix has rank exactly 2.

Solution. This argument can be completed by writing out a 3×3 matrix and explicitly examining the entries. Consider any skew-symmetric 3×3 matrix

$$A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}.$$

Since 3 is odd, then the previous problem implies that $\det A = 0$ so that $\mathrm{rk}(A) < 3$. Since $A \neq 0$, then one of a, b, c is not 0. Now we can examine each case.

If $a \neq 0$, then the first and second columns are nonzero and independent, so the rank is 2.

If $b \neq 0$, then the first and third columns are nonzero and independent, so the rank is 2.

If $c \neq 0$, then the second and third columns are nonzero and independent, so the rank is 2.