Homework 4 Solutions
February 1, 2020

5.20.4. Given a real orthogonal matrix A with -1 as an eigenvalue of multiplicity k, prove that $\det A = (-1)^k$.

Solution. Since A is orthogonal, then for all eigenvalues λ, we have $|\lambda| = 1$. Since A is real, the complex eigenvalues come in conjugate pairs, and the determinant is calculated as follows

$$\det A = \prod_i \lambda_i = (1) \prod_{1 \le i \le k} -1 = (-1)^k$$

5.20.6. If T is both unitary and Hermitian, prove that $T^2 = I$.

Solution. On the one hand, since T is unitary, then $\langle T(x), T(y)\rangle = \langle x, y\rangle$. On the other hand since T is Hermitian, then $\langle T(x), y\rangle = \langle x, T(y)\rangle$. Then for all x, y,

$$\langle T^2(x), y\rangle = \langle T(x), T(y)\rangle = \langle x, y\rangle.$$

Since $T^2(x)$ and x have all the same inner products, $T^2(x) = x$ for all x so $T^2 = I$.

8.3.1abd. Make a sketch of the level sets of some functions. (a) $f(x, y) = x^2 + y^2$ (b) $f(x, y) = e^{xy}$ (d) $f(x, y, z) = x + y + z$

Solution. I'm not going to tex up a drawing so let me know if you want one. (a) $x^2 + y^2 = c$ is a circle of radius $\sqrt{c}$. (b) $c = e^{xy}$ is empty for $c \le 0$ and looks like a hyperbola for $c > 0$. (d) You get planes with normal $(1, 1, 1)$ going through the point $(c, 0, 0)$.

2acl. Determine whether the following sets are open or not by drawing a sketch. (a) $x^2 + y^2 < 1$ (c) $|x|, |y| < 1$ (l) $|x| < 2$, $y > x^2$

Solution. I'm not going to tex up a drawing so let me know if you want one. (a) This set is open, it is the open unit circle. (c) This set is open, it is an infinite box. (l) This set is open, it is the intersection of an open column with the open set above the parabola.

8.3.3bd. Determine whether following sets are open or not. (b) $|x|, |y|, |z| < 1$ (d) $|x| \le 1, |y|, |z| < 1$.

Solution. (b) This set is open. For any (x, y, z) in S, pick $r < \min\{|x-1|, |y-1|, |z-1|\}$. (d) This set is not open. If $x = 1$, for example $(1, 0, 0)$, then no ball around $(1, 0, 0)$ is contained in S, the point $(1 + r/2, 0, 0)$ will be in any ball.

8.3.4. (a) If A is an open set, and if $x \in A$, show that $A - \{x\}$ is open. (b) If $A = (a, b) \subset \mathbb{R}^1$, and $B = [c, d]$ is a closed subinterval, show that $A - B$ is open.

Solution. (a) Let $y \in A - \{x\}$. Since $y \in A$, there exist a ball $B_r(y) \subset A$. Let $|y - x| = r'$. Pick $r'' < \min\{r, r'\}$. Then by construction $B_{r''}(y) \subset A - \{x\}$ as desired. (b) Similarly, for $y \in A - B$, then there exists a ball $B_r(y) \subset A$. Pick $r' = \min\{|y - c|, |y - d|, r\} > 0$. Then $B_{r'}(y) \subset A - B$ as desired.

8.3.9. Let $S \subseteq \mathbb{R}^n$, with interior I and exterior E. Prove that I and E are open. (b) Show that $\mathbb{R}^n = I \cup \partial S \cup E$ is a union of disjoint sets, and deduce that ∂S is always closed.

Solution. (a) Let $s \in I$. By definition, there exists a ball $B_r(s)$ such that all $t \in B_r(s)$ are also in S. We show that $t \in I$. Let $||t - s|| = r'$, and let $r_0 < r - r'$. Then $B_{r_0}(t) \subset B_r(s)$. This implies that for all $t' \in B_{r'}(t)$, then $t' \in B_r(S)$ which by definition implies that $t' \in S$. Thus $t \in I$ also by definition.

Similarly for E. Let $s \in E$, so that there is a ball $B_r(s)$ which contains no points of s. Then for any $t \in B_r(s)$, we show that $t \in E$. Again construct $B_{r_0}(t)$ inside of $B_r(s)$. No point of this ball around t is in S so $t \in E$.

(b) By definition, $\partial S = (I \cup E)^c$ so it suffices to show that $E \cap I = \emptyset$. This is trivial by defintion. If $s \in I \cap E$, then there is an open ball contained entirely in S and also contains no points of S. As long as S is nonempty, this is a contradiction. So I and E are disjoint. Then $\mathbb{R}^n = I \cup \partial S \cup E$ is a disjoint union.

Since $\partial S = (I \cup E)^c$ and a union of open sets is open, then ∂S is closed.

8.5.2. Let the limit of f as $(x, y) \to (a, b)$ exist in $\mathbb{R}^2$, and the denote the limit by L. Assume $\lim_{x\to a} f(x, y)$ and $\lim_{y\to b} f(x, y)$ exist. Show that the iterated limit exist and are both L.

Solution. Let $g(x) = \lim_{y\to b} f(x, y)$.

By definition, we must show that there exists a radius r such that for all $x \in B_r(a)$ that $|g(x) - L| < \varepsilon$. It is in fact equivalent to show that for all sequences such that $x_i \to a$, then $g(x_i) \to L$. This is equivalence has a short proof. One the one hand if the limit exists in the usual sense, then $g(x_i) \to L$ by definition of the limit of a sequence. On the other hand, if the limit doesn't exist, then for all $r > 0$, we can find an $x \in B_r(a)$ such that $|g(x) - L| > \varepsilon$. Let $r = 1/n$, let the corresponding $x = x_n$. Then the sequence $x_n \to a$ by construction, but $g(x_i) \not\to L$. By contrapositive, the two definitions are equivalent. This equivalence holds in general, so we can apply it the limit in $\mathbb{R}^2$ as well.

So, we use this second definition of convergence to solve the problem. Let $x_n \to a$. For each x_n, pick a y_n such that $|y_n - b| < 1/n$ and such that $|f(x_n, y_n) - g(x_n)| < \varepsilon/2$ since the limit of f as $y \to b$ exists. Consider the sequence (x_i, y_i), which converges to (a, b). Since this sequence converges to (a, b), then $f(x_n, y_n) \to L$ by equivalence above.

Now fix an N such that for all $n \geq N$ we have $f(x_n, y_n) - L| < \varepsilon/2$. Then in this case

$$|g(x_n) - L| \leq |g(x_n) - f(x_n, y_n)| + |f(x_n, y_n) - L| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus $g(x_n) \to L$ as desired. The other iterated limit follows by the symmetric argument.

8.5.4. Let $f(x, y) = (x^2y^2)/(x^2y^2+(x-y^2))$ whenever the denominator doesn't vanish. Show that the iterated limits both are 0, but that $f \not\to 0$ as $(x, y) \to 0$.

Solution. For $x \neq 0$, we can take $\lim_{y\to 0} f(x, y)$ and simply plug in $y = 0$, to get that the limit is 0. Then the limit as $x \to 0$ is also 0. A similar argument for the other order holds. Now, we show that the limit in $\mathbb{R}^2$ doesn't exist. On the one hand, if we approach along the line $y = 0$, we see that $f(x, 0) = 0$ in a neighborhood around the orgin. On the other hand if $y = x$, then we see that $f(x, x) = 1$ in a neighborhood around the origin. Therefore there exists no $r > 0$ such that for $||x - 0|| < r$, that we have $|f(x, y) - L| < \varepsilon$ for any L. In particular $|f(x, 0) - f(x, x)| = 1$ arbitrarily close to the origin. Thus the limit doesn't exist in $\mathbb{R}^2$.

8.5.6. Let $f(x, y) = (x^2 - y^2)/(x^2 + y^2)$. Find the limit of f as $(x, y) \to 0$ along $y = mx$. Is it possible to define $f(0, 0)$ so that f is continuous there?

Solution. Note that $f(x, mx) = (x^2 - m^2x^2)/(x^2 + m^2x^2) = (1 - m^2)/(1 + m^2)$. Since f has constant value along this line, we get that this expression is the limit too. Since the limit depends on m, then we can conclude that the limit of f as $(x, y) \to 0$ does not exist in $\mathbb{R}^2$. Therefore there is no way to define $f(0, 0)$ to make f continuous at the origin by definition of continuity.

8.9.1. A scalar field f is defined on $\mathbb{R}^n$ by the equation $f(x) = a \cdot x$ where a is a constant vector. Compute $f'(x; y)$.

Solution. Note that $f(x) = \sum_i a_i x_i$. Then each partial exists and is continuos since it is constant. Therefore f is differentiable and thus $f'(x; y)$ exists and has the formula $f'(x; y) = \nabla f(x) \cdot y$. Computing this out, we obtain

$$f'(x; y) = \nabla f(x) \cdot y = \sum_i a_i y_i = a \cdot y.$$

8.9.3. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Compute the derivative $f'(x; y)$ for the scalar field $f(x) = x \cdot T(x)$.

Solution. Fix the standard basis and let $T = A$. Then $f(\vec{x}) = \langle x, Ax \rangle$, which has the written out expression $f(\vec{x}) = \sum_{i,j} a_{ij} x_i x_j$. Then each partial exists and is continuous (its linear), so therefore f is differentiable and all the directional derivatives exist. Writing everything out, I got something like this.

$$f'(x; y) = \nabla f(x) \cdot y = \sum_j y_j \left(\left(\sum_{i \neq j} (a_{ij} + a_{ji}) x_i \right) + 2a_{jj} x_j \right)$$

8.9.4,5,8,9. Compute the first order partials of the following functions. (4) $f(x, y) = x^2 + y^2 \sin(xy)$ (5) $f(x, y) = \sqrt{x^2 + y^2}$ (8) $f(\vec{x}) = a\vec{x}$ (9) $f(\vec{x}) = \sum_{i,j} a_{ij} x_i x_j$ where $a_{ij} = a_{ji}$.

Solution. (4) $\nabla f = (2x + y^3 \cos(xy), 2y \sin(xy) + xy^2 \cos(xy))$ (5) $\nabla f(v) = v/f(v)$ where $v = (x, y)$ (8) $D_i(f) = a_i = a \cdot e_i$ (9) $D_k(f) = 2 \sum_i a_{ki} x_i$

8.9.10. Compute the partials of $f(x, y) = x^4 + y^4 - 4x^2y^2$ and show the mixed partials are equal.

Solution. $D_x(f) = 4x^3 - 8xy^2$, $D_y(f) = 4y^3 - 8x^2y$, and $D_{xy}(f) = D_{yx}(f) = -16xy$.

8.9.18. Let $v(r, t) = t^n e^{-r^2/(4t)}$. Find a value of the constant n such that v satisfies the following equation $D_t(v) = \frac{1}{r^2} D_r(r^2 D_r(v))$.

Solution. The left hand side is $4D_t(v) = t^{n-2} e^{-r^2/(4t)}(4nt + r^2)$. The right hand side is $\frac{1}{4} t^{n-2} e^{-r^2/(4t)}(r^2 - 6t)$. When equation the sides, most terms cancel and the equation simplifies to $4n = -6$ which implies $n = -3/2$.

8.14.1ace. Find the gradient vector at each point at which it exists for the scalar fields defined by the following equations. (a) $f(x, y) = x^2 + y^2 \sin(xy)$ (c) $f(x, y, z) = x^2y^3z^4$ (e) $f(x, y, z) = \log(x^2 + 2y^2 - 3z^2)$.

Solution. (a) This is a composition, product, and sum of globally differentiable functions, and thus has gradient everywhere.

$$\nabla f = (2x + y^3 \cos(xy), 2y \sin(xy) + xy^2 \cos(xy))$$

(c) Similarly, this is differentiable everywhere and

$$\nabla f = (2xy^3z^4, 3x^2y^2z^4, 4x^2y^3z^3).$$

(e) Since log is differentiable only for positive reals, then we must restrict the open set where $x^2 + 2y^2 - 3z^2 > 0$. On this domain we have that

$$\nabla f(v) = 2v/f(v)$$

where $v = (x, y, z)$.

8.14.3. Find the points (x, y) and the directions for which the directional derivative of $f(x, y) = 3x^2 + y^2$ has its largest value, if (x, y) is on the unit circle.

Solution. We can maximize the directional derivative at each point on the unit circle, and then find the points that achieve the maximum of that function.

The maximum directional derivative at a point x is achieved by the direction $u = \frac{\nabla f}{||\nabla f||}$. In this case the angle between the direction and the gradient is 0, therefore the dot product is maximized. In this case

$$D_{u_{\max}}(f) = \nabla f \cdot \frac{\nabla f}{||\nabla f||} = ||\nabla f||.$$

For $f(x, y) = 3x^2 + y^2$, we obtain

$$D_{u_{\max}}(f) = ||(6x, 2y)|| = \sqrt{36x^2 + 4y^2} = 2\sqrt{9x^2 + y^2}.$$

Thus we must maximize this function on the unit circle.

We parametrize the unit circle by $(\cos(t), \sin(t))$, so and rewrite f as $g(t) = 2\sqrt{9\cos^2(t) + \sin^2(t)} = 2\sqrt{8\cos^2(t) + 1}$. We obtain the critical points by solving $g'(t) = 0$. The numerator of the derivative is $\sin(t)\cos(t)$, so it suffices to solve $\sin(t)\cos(t) = 0$. This occurs at $t = \frac{k\pi}{2}$ for $k \in \mathbb{Z}$. Taking the double derivative, we obtain maxima at $t = k\pi$ for $k \in \mathbb{Z}$, i.e. angles of 0 and π. This corresponds to the points $(1, 0)$ and $(-1, 0)$. The unit vector which gives the maximum directional derivative at the points is also $(1, 0)$ and $(-1, 0)$ by the above formula.

8.14.4. A differentiable scalar field f has at the points $(1, 2)$ directional derivatives 2 in the direction toward $(2, 2)$ and -2 in the direction toward $(1, 1)$. Determine the gradient vector at $(1, 2)$ and compute the directional derivative in the direction toward $(4, 6)$.

Solution. Note that $(2, 2) - (1, 2) = (1, 0)$ and $(1, 1) - (1, 2) = -(0, 1)$. Therefore the assumptions are the problem is that $D_1(f) = 2$ and $D_2(f) = 2$. Therefore the gradient is $(2, 2)$. Furthermore the directional derivative toward $(4, 6)$ is the directional derivative of the vector

$$u = \frac{v}{||v||}$$

where $v = (4, 6) - (1, 2) = (3, 4)$. Therefore $u = (3/5, 4/5)$ and $D_u(f) = (2, 2) \cdot u = 14/5$.

8.14.9. Assume f is differentiable at each point of an n-ball $B(a)$. If $f'(x; y) = 0$ for n independent vectors $y_1, \ldots, y_n$ for every x in $B(a)$, prove that f is constant on $B(a)$.

Solution. It suffices to show that $Df = 0$ for all $x \in B(a)$. Since f is differentiable, then $f'(x; y) = \nabla f \cdot y$ exists for all y. Write $y = \sum_i a_i y_i$. Therefore,

$$f'(x; y) = \nabla f \cdot y = \sum a_{\nabla} f \cdot y_i = \sum a_i f'(x; y_i) = 0.$$

If $y = e_i$, we see that $\frac{\partial f}{\partial x_i} = 0$ for all i. Thus f is constant on $B(a)$.