Homework 4 Solutions
February 24, 2019

8.14.10. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be differentiable in an n-ball $B(a)$. Show that (a) if $\nabla f = 0$ for every $x \in B(a)$, then f is constant on $B(a)$. (b) If $f(x) \leq f(a)$ for all $x \in B(a)$, then $\nabla f(a) = 0$.

Solution. (a) Let $x \in B(a)$. We show that $f(x) = f(a)$. Since f is differentiable, then $f'(a; v)$ exists. Then by the MVT for derivatives of scalar fields, Theorem 8.4, we have that there exists a $t \in [0, 1]$ such that $f(a + (x - a)) - f(a) = f'(a + t(x - a); (x - a))$. Since $a + t(x - a) \in B(a)$, then the RHS is 0, so that $f(a) = f(x)$.

(b) We show that $f'(a; e_i) = 0$ for all i. By definition,

$$f'(a; e_i) = \lim_{h \to 0} \frac{f(a + he_i) - f(a)}{h}.$$

Note that for sufficiently small h, the vector $a + he_i \in B(a)$ so that $f(a + he_i) \leq f(a)$. Therefore this is a limit of nonpositive values and if the limit exists, it must be a nonpositive as well. Since f is differentiable at a, this limit exists so that $f'(a; e_i) \leq 0$. Similarly, $-f'(a; e_i) = \lim_{h\to 0} \frac{f(a)-f(a+he_i)}{h}$, which by a similar argument must also be ≤ 0. Therefore $f'(a; e_i) = 0$ and $\nabla f(a) = 0$.

8.17.1ab. Let $u = f(x, y)$. Set $x = x(t)$ and $y = y(t)$. Then $u = F(t)$. (a) Use the chain rule to compute F'. (b) Similarly, compute $F''(t)$.

Solution. (a) Let $T(t) = (x(t), y(t)$ so that $F = f \circ T$. By the chain rule

$$DF = Df(T(t))DT(t) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \frac{\partial f}{\partial x} x'(t) + \frac{\partial f}{\partial y} y'(t).$$

(b) To compute the double derivative, it suffices to compute $D(D_1 f)$ and $D(D_2 f)$. By similar calculations to above:

$$D\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} x' + \frac{\partial^2 f}{\partial x \partial y} y'$$
$$D\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} x' + \frac{\partial^2 f}{\partial y^2} y'$$

Now by the product rule:

$$\begin{aligned} F''(t) &= D(D_1 f x') + D(D_2 f y') \\ &= D(D_1 f)x' + x'' D_1 f + y'' D_2 f + y' D(D_2 f) \\ &= \left(\frac{\partial^2 f}{\partial x^2} x' + \frac{\partial^2 f}{\partial x \partial y} y' \right) x' + x'' \frac{\partial f}{\partial x} + y'' \frac{\partial f}{\partial y} + y' \left(\frac{\partial^2 f}{\partial x \partial y} x' + \frac{\partial^2 f}{\partial y^2} y' \right) \\ &= x'' \frac{\partial f}{\partial x} + \frac{\partial^2 f}{\partial x^2} (x')^2 + 2x'y' \frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2} (y')^2 + y'' \frac{\partial f}{\partial y} \end{aligned}$$

8.17.2a. Use the previous exercise to compute the derivative of $f(x(t), y(t)) = x(t)^2 + y(t)^2 = t^2 + t^4$.

Solution. $F'(t) = (2t)(1) + (2t^2)(2t) = 2t + 4t^3$ and $F''(t) = (0)(2t) + (2)(1)^2 + 2(1)(2t)(0) + (2)(2t)^2 + (2)(2t^2) = 2 + 12t^2$

8.17.3. Evaluate some directional derivatives.

Solution. (a) Note that on a sphere, the normal vector is just the vector itseslf. Thus

$$f'((2,2,1);(2,2,1)) = (3,-5,2) \cdot (2,2,1) = 6 - 10 + 2 = -2.$$

If we require the vector be a unit vector, divide it by 3.

(b) Similarly, for v on the sphere of radius 2, we can compute $f'(v; v) = (2x, -2y, 0) \cdot v = 2(x^2 - y^2)$. If the normal vector needs to be a unit vector, then we obtain $f'(v; v/2) = x^2 - y^2$.

(c) First we find the intersection of the two curves. If $z^2 = x^2 + y^2$ and $z^2 = 2x^2 + 2y^2 - 25$ then on the intersection, we have $x^2 + y^2 = 25$, which is circle of radius 5, and therefore $z = 5$ as well. Thus a parametrization of this intersection is $\varphi(t) = (5\cos(t), 5\sin(t), 5)$. The tangent vector is $\varphi'(t) = (-5\sin(t), 5\cos(t), 0)$, and at $(3,4,5)$, we obtain that the tangent vector is $v = (-4, 3, 0)$. Normalizing, $v = (-4/5, 3/5, 0)$. Now

$$f'((3,4,5);(-4/5,3/5,0)) = (6,8,-10) \cdot (-4/5,3/5,0) = 0.$$

8.17.4. (a) Find a vector $V(x,y,z)$ normal to the surface $z = ||(x,y,0)|| + ||(x,y,0)||^3$. (b) Find the cosine of the angle θ between V and the z-axis and determine the limit $\cos(\theta)$ as $(x,y,z) \to (0,0,0)$.

Solution. The normal in this case is $n = (-D_x f, -D_y f, 1)$ since we are on the graph of a scalar field. Therefore

$$n = V(x,y,z) = -(\frac{x}{\sqrt{x^2+y^2}} + 3x\sqrt{x^2+y^2}, \frac{y}{\sqrt{x^2+y^2}} + 3y\sqrt{x^2+y^2}, -1).$$

The cosine of the angle is $\cos(\theta) = \frac{v \cdot w}{||v|| ||w||}$, so that

$$\lim_{(x,y,z)\to(0,0,0)} \cos(\theta) = \lim_{x,y\to(0,0)} \frac{1}{||V||} = \lim_{(x,y)\to(0,0)} \frac{1}{\sqrt{1+(1+3(x^2+y^2)^2)}} = \frac{1}{\sqrt{2}}.$$

8.17.6. Let $f(x,y) = \sqrt{|xy|}$. (a) Show that $D_x f(0,0) = D_y f(0,0) = 0$ and (b) determine whether the surface $z = f(x,y)$ has a tangent plane at the origin.

Solution. (a) By definition $f'(0;(x,y)) = \lim_{h\to 0} \frac{\sqrt{|h^2xy|h}}{=} xy$. For $(x,y) = e_i$, we see that the limit is 0.

(b) To show that $z = f(x,y)$ has no tangent plane, we show that f is not differentiable at 0. Indeed if we consider the directional derivative along $(1,1)$, then

$$f'(0;(1,1)) = \lim_{h\to 0} \frac{\sqrt{|h^2|}}{h}$$

which does not exist (since h can be positive or negative). In particular $f'(0;(1,1)) \neq \nabla f \cdot (1,1)$. Thus f is not differentiable at 0 so there is no tangent plane.

8.17.12. If ∇f is always parallel to (x,y,z) show that f must assume equal values at the points $(0,0,a)$ and $(0,0,-a)$.

Solution. Acknowledgments to Cameron for bringing this solution to my attention.

Let $\varphi(t) = (0, a\sin(t), a\cos(t))$ for $t \in [-1,1]$. Then by the chain rule,

$$D(f\circ\varphi) = Df(\varphi(t))D\varphi(t) = \nabla f(\varphi(t)) \begin{bmatrix} 0 \\ a\cos(t) \\ -a\sin(t) \end{bmatrix}.$$

But $\nabla f(\varphi(t)) = \lambda(t)\varphi(t)$, so that

$$D(f\circ\varphi) = \lambda(t)\varphi\cdot\varphi' = 0.$$

Therefore f has constant value on φ, which implies that f is equivalued on $(0,0,a)$ and $(0,0,-a)$.

8.22.1. Let $t = g(x,y)$ so that $F(t) = f(x,y)$, i.e. $f = F\circ g$. (a) Compute Df in general. (b) Plug in $F(t) = e^{\sin(t)}$ and $g(x,y) = \cos(x^2+y^2)$ and see that your formula works.

Solution. (a) Since $f = F \circ g$, by the chain rule

$$Df = DF(g(x,y))Dg = \begin{bmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} F'(g(x,y))\frac{\partial g}{\partial x} & F'(g(x,y))\frac{\partial g}{\partial y} \end{bmatrix}.$$

(b) If $F(t) = e^{\sin(t)}$ and $t = \cos(x^2 + y^2)$, then $f(x,y) = e^{\sin(\cos(x^2+y^2))}$. Naive chain rule says that

$$\begin{aligned} \frac{\partial f}{\partial x} &= -2x\sin(x^2+y^2)\cos(\cos(x^2+y^2))e^{\sin(\cos(x^2+y^2))} \\ \frac{\partial f}{\partial y} &= -2y\sin(x^2+y^2)\cos(\cos(x^2+y^2))e^{\sin(\cos(x^2+y^2))} \end{aligned}$$

which agrees with the above formula.

8.22.2. Let $f(u,v)$ be a scalar field, let $u = (x-y)/2$, $v = (x+y)/2$, so that $f(u,v) = F(x,y)$. Find DF.

Solution. Let $T(x,y) = (x-y, x+y)/2$ so that $F = f \circ T$. Then

$$\begin{aligned} DF = Df(T(x,y))DT(x,y) &= \frac{1}{2}\begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} \frac{\partial f}{\partial u} + \frac{\partial f}{\partial v} & \frac{\partial f}{\partial u} - \frac{\partial f}{\partial v} \end{bmatrix} \end{aligned}$$

8.22.3. Let $u = f(x,y)$ and let $x = X(s,t)$, $y = Y(s,t)$, so that $u = F(s,t) = f(X(s,t), Y(s,t))$. (a) Compute DF. (b) Compute the double derivatives of F.

Solution. (a) Let $T(s,t) = (X(s,t), Y(s,t))$ so that $F = f \circ T$. By the chain rule

$$\begin{aligned} DF(s,t) &= Df(T(s,t))DT(s,t) \\ &= \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial X}{\partial s} & \frac{\partial X}{\partial t} \\ \frac{\partial Y}{\partial s} & \frac{\partial Y}{\partial t} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial f}{\partial x}\frac{\partial X}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial Y}{\partial s} & \frac{\partial f}{\partial x}\frac{\partial X}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial Y}{\partial t} \end{bmatrix}. \end{aligned}$$

Each component is the desired partial.

(b) To find the mixed partials, we can consider for example

$$D(D_sF) = D\left(\frac{\partial f}{\partial x}\frac{\partial X}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial Y}{\partial s}\right).$$

One can plug in this function for f in the previous formula, yielding the desired formula in the book. For example,

$$\begin{aligned}
\frac{\partial^2 F}{\partial s^2} &= \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial s}\right)\frac{\partial X}{\partial s} + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial t}\right)\frac{\partial Y}{\partial s} \\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\frac{\partial X}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial Y}{\partial s}\right)\frac{\partial X}{\partial s} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\frac{\partial X}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial Y}{\partial t}\right)\frac{\partial Y}{\partial s} \\
&= \frac{\partial f}{\partial x}\frac{\partial^2 X}{\partial s^2} + \frac{\partial^2 f}{\partial x^2}\left(\frac{\partial X}{\partial s}\right)^2 + 2\frac{\partial X}{\partial s}\frac{\partial Y}{\partial s}\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial f}{\partial y}\frac{\partial^2 Y}{\partial s^2} + \frac{\partial^2 f}{\partial y^2}\left(\frac{\partial Y}{\partial s}\right)^2.
\end{aligned}$$

(c) The other two formulas are similar.

8.22.5. Let $\varphi(r,\theta) = f(r\cos(\theta), r\sin(\theta))$ for a scalar field $f\colon \mathbb{R}^2 \to \mathbb{R}$. Express the double derivative of φ in terms of the double derivatives of $f(x,y)$.

Solution. (Apologies for the switching in notations. Had to do with copy-pasting.) By Example 8.21, we have

$$\tfrac{\partial \varphi}{\partial r} = \tfrac{\partial f}{\partial x}\cos(\theta) + \tfrac{\partial f}{\partial y}\sin(\theta) \quad \tfrac{\partial \varphi}{\partial \theta} = -r\tfrac{\partial f}{\partial x}\sin(\theta) + r\tfrac{\partial f}{\partial y}\cos(\theta)$$

To compute the double derivatives, it suffices to compute $D_r(D_x f)$, $D_\theta(D_x f)$, $D_r(D_y f)$, and $D_\theta(D_y f)$ and apply the product rule ad nauseum. By the above rules applied to $D_x f$ and $D_y f$ we obtain

$$\begin{aligned}
\frac{\partial}{\partial r}\frac{\partial f}{\partial x} &= \frac{\partial^2 f}{\partial x^2}\cos(\theta) + \frac{\partial^2 f}{\partial x \partial y}\sin(\theta) \\
\frac{\partial}{\partial \theta}\frac{\partial f}{\partial x} &= -r\frac{\partial^2 f}{\partial x^2}\sin(\theta) + r\frac{\partial^2 f}{\partial x \partial y}\cos(\theta) \\
\frac{\partial}{\partial r}\frac{\partial f}{\partial y} &= \frac{\partial^2 f}{\partial x \partial y}\cos(\theta) + \frac{\partial^2 f}{\partial y^2}\sin(\theta) \\
\frac{\partial}{\partial \theta}\frac{\partial f}{\partial y} &= -r\frac{\partial^2 f}{\partial x \partial y}\sin(\theta) + r\frac{\partial^2 f}{\partial y^2}\cos(\theta).
\end{aligned}$$

Now,

$$\begin{aligned}
D_r^2\varphi &= D_r(D_x f\cos(\theta)) + D_r(D_y f\sin(\theta)) \\
&= D_r D_x f\cos(\theta) + D_r D_y f\sin(\theta) \\
D_r D_\theta \varphi &= D_r(-rD_x f\sin(\theta) + rD_y f\cos(\theta)) \\
&= -D_x f\sin(\theta) + D_r D_x f\sin(\theta) + D_y f\cos(\theta) + rD_r D_y f\cos(\theta) \\
D_\theta^2\varphi &= D_\theta(-rD_x f\sin(\theta) + rD_y f\cos(\theta)) \\
&= -rD_\theta D_x f\sin(\theta) - rD_x f\cos(\theta) + rD_\theta D_y f\cos(\theta) - rD_y f\sin(\theta)
\end{aligned}$$

where the mixed polar and cartesian derivatives in terms of the derivatives of f are listed above.

8.22.14. Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ and $g\colon \mathbb{R}^3 \to \mathbb{R}^2$ be $f(x,y) = (e^{x+2y}, \sin(y+2x))$ and $g(u,v,w) = (u + 2v^2 + 3w^3, 2v - u^2)$. (a) Compute Df and Dg. (b) Compute the composition $h(u,v,w) = f(g(u,v,w))$. (c) Compute the Jacobian Dh.

Solution. (a)

$$Df = \begin{bmatrix} e^{x+2y} & 2\cos(2x+y) \\ 2e^{x+2y} & \cos(2x+y) \end{bmatrix} \quad Dg = \begin{bmatrix} 1 & 4v & 9w^2 \\ -2u & 2 & 0 \end{bmatrix}$$

(b)

$$\begin{aligned} h(u,v,w) &= f(g(u,v,w)) \\ &= f(u + 2v^2 + 3w^3, 2v - u^2) \\ &= (e^{u+2v^2+3w^3+4v-2u^2}, \sin(2v - u^2 + 2u + 4v^2 + 6w^3)) \end{aligned}$$

(c)

$$Dh(u,v,w) = Df(g(u,v,w))Dg(u,v,w) = \begin{bmatrix} A - 4uB & 4A - 2vB & 9w^2A \\ 2A - 2uB & 8A - vB & 18w^2A \end{bmatrix}$$

where $A = e^{u+2v^2+3w^3+4v-2u^2}$ and $B = \cos(2v - u^2 + 2u + 4v^2 + 6w^3)$.

8.22.15. Let $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ and $g\colon \mathbb{R}^3 \to \mathbb{R}^3$ be $f(x,y,z) = (x^2 + y + z, 2x + y + z^2)$ and $g(u,v,w) = (uv^2w^2, w^2\sin(v), u^2e^v)$. (a) Compute Df and Dg. (b) Compute the composition $h(u,v,w) = f(g(u,v,w))$. (c) Compute the Jacobian Dh.

Solution. (a)

$$Df = \begin{bmatrix} 2x & 1 & 1 \\ 2 & 1 & 2z \end{bmatrix} \quad Dg = \begin{bmatrix} v^2w^2 & 2uvw^2 & 2uv^2w \\ 0 & w^2\cos(v) & 2w\sin(v) \\ 2ue^v & u^2e^v & 0 \end{bmatrix}$$

(b)

$$\begin{aligned} h(u,v,w) &= f(g(u,v,w)) \\ &= f(uv^2w^2, w^2\sin(v), u^2e^v) \\ &= (u^2v^4w^4 + w^2\sin(v) + u^2e^v, 2uv^2w^2 + w^2\sin(v) + u^4e^{2v}) \end{aligned}$$

(c)

$$\begin{aligned} Dh(u,v,w) &= Df(g(u,v,w))Dg(u,v,w) \\ &= \begin{bmatrix} 2uv^4w^4 + 2ue^v & 4u^2v^3w^4 + w^2\cos(v) + u^2e^v & 4u^2v^4w^3 + 2w\sin(v) \\ 2v^2w^2 + 4u^3e^{2v} & 4uvw^2 + w^2\cos(v) + 2u^4e^{2v} & 4uv^2w + 2w\sin(v) \end{bmatrix} \end{aligned}$$

8.24.1. Find a scalar field f satisfying both the following conditions: (a) the partial derivatives exist and are 0 (b) the directional derivative at the origin in the direction $(1,1)$ exists and has the value 3. Explain why such an f cannot be differentiable at the origin.

Solution. Let $f(x,y) = \frac{6xy}{x+y}$ and $f(0,0) = 0$. Then by definition, the directional derivative

$$f'(0;v) = \lim_{h\to 0} \frac{6h^2 v_1 v_2}{h^2 v_1 + h^2 v_2} = \frac{6v_1 v_2}{v_1 + v_2}.$$

If $v = (1,0)$ or $(0,1)$, then $f'(0;v) = 0$ as desired. If $v = (1,1)$, then $f'(0;v) = 6/2 = 3$ as desired.

This function cannot be differentiable at the origin, since $f'(0;(1,1)) = 3 \neq (D_1 f(0), D_2 f(0)) \cdot (1,1) = 0$.

8.24.3. Let $f(x,y) = xy^3/(x^3+y^6)$ if $(x,y) \neq (0,0)$ and $f(0,0) = 0$. (a) Prove that the derivative $f'(0;a)$ exists for all a and compute its value. (b) Determine whether or not f is continuous at the origin.

Solution. (a) Let $a = (a_1, a_2)$. By definition of the directional derivative,

$$\begin{aligned} f'(0;a) &= \lim_{h\to 0} \frac{(ha_1)(ha_2)^3}{h((ha_1)^3 + (ha_2)^6)} \\ &= \lim_{h\to 0} \frac{h^4 a_1 a_2}{h^4(a_1^3 + h^3 a_2^6)} \\ &= \frac{a_1 a_2}{a_1^3} = \frac{a_2}{a_1^2}. \end{aligned}$$

This holds for $a_1 \neq 0$. If $a_1 = 0$, then it is clear from the expression that $f'(0;a) = 0$.

(b) However, despite every directional derivative existing, the function is not continuous at the origin. Let $(x,y) = (x,x)$. Then

$$\lim_{x\to 0} f(x,x) = \lim_{x\to 0} \frac{x^4}{x^3 + x^6} = \lim_{x\to 0} \frac{x}{1+x^3} = 0.$$

On the other hand, if we let $x = y^2$, then

$$\lim_{y\to 0} f(x,y) = \lim_{y\to 0} \frac{y^5}{y^6 + y^6} = \frac{1}{2} \lim_{y\to 0} \frac{1}{y}$$

which does not exist. Therefore f is not continuous at the origin.

8.24.12. Let $R = (x, y, z)$, let $r = ||R||$. If A and B are constant vectors, show that (a) $A \cdot \nabla(1/r) = -(A \cdot R)/r^3$ (b) $B \cdot \nabla\left(A \cdot \nabla(1/r)\right) = 3(A \cdot r)(B \cdot r)/r^5 - (A \cdot B)/r^3$

Solution. (a) This equality can be seen by the usual rules of the derivative.

$$A \cdot \nabla\left(\frac{1}{r}\right) = (A \cdot 2R)\frac{-1}{2}\frac{1}{(x^2+y^2+z^2)^{3/2}} = -\frac{A \cdot R}{r^3}$$

(b) After telling you in class to write it out, I then did part (a) and realized you could apply the quotient rule and use part (a).

$$\begin{aligned} B \cdot \nabla\left(A \cdot \nabla(1/r)\right) &= B \cdot \nabla\left(\frac{-A \cdot R}{r^3}\right) \\ &= B \cdot \frac{\nabla(-A \cdot R)r^3 - (-A \cdot R)\nabla(r^3)}{r^6} \\ &= B \cdot \frac{-Ar^3 - (-A \cdot R)3Rr}{r^6} \\ &= \frac{3(A \cdot R)(B \cdot R)}{r^5} - \frac{A \cdot B}{r^3} \end{aligned}$$

Additional Problem 1.. A function $G\colon \mathbb{R}^n \to \mathbb{R}$ is called homogeneous of degree 1 if $G(tx) = tG(x)$ for all $t > 0$ and all $x \neq 0 \in \mathbb{R}^n$. Suppose G is homogeneous of degree 1 and continuous. (a) Show that $G(0) = 0$. (b) Show that the directional derivative of G at 0 along y exists for all $y \in \mathbb{R}^n$. (c) Show that G is differentiable at 0 iff it is linear.

Solution. (a) Since $G(tx) = tG(x)$ for all $t > 0$ and $||x|| > 0$, we can fix $x \neq 0$ and consider

$$\lim_{t \to 0} G(tx) = \lim_{t \to 0} tG(x).$$

Taking the limit on the left hand side, we obtain that the limit is 0 since $G(x)$ is constant with respect to t. Taking the limit on the right side, by continuity of G, we get $G(0)$.

(b) Let $v \neq 0$, then we show that $G'(0; v)$ exists by definition.

$$
\begin{aligned}
G'(0;v) &= \lim_{h\to 0} \frac{G(0+hv)-G(0)}{h} \\
&= \lim_{h\to 0} \frac{G(hv)}{h} \\
&= \lim_{h\to 0} \frac{hG(v)}{h} \\
&= G(v)
\end{aligned}
$$

Hence we conclude $G'(0;v) = G(v)$.

(c) Assume G is differentiable. Then $G(v) = G'(0;v) = DG(0)v$, which is a linear function . If G is linear, then it is differentiable trivially.