Homework 6 Solutions
February 1, 2020

8.24.13. Find the set of points (a, b, c) such that $(x-a)^2+(y-b)^2+(z-c)^2 = 1$ intersects $x^2 + y^2 + z^2 = 1$ orthogonally.

Solution. By intersect orthogonally, the two tangent planes must be orthgonal, i.e. the normal vectors are orthogonal. Note that $n = (x, y, z)$ and $n' = (x - a, y - b, z - c)$ for each sphere.

Let (x, y, z) lie on the intersection of the two spheres. By the above discussion, to find (a, b, c), we also must satisfy the additional equation

$$n \cdot n' = x(x - a) + y(y - b) + z(z - c) = 0.$$

Expanding and plugging in $x^2 + y^2 + z^2 = 1$, we obtain the relation

$$ax + by + cz = 1.$$

On the other hand, if one expands the equation for the other sphere, then

$$(x-a)^2+(y-b)^2+(c-z)^2 = x^2+y^2+z^2-2(ax+by+cz)+(a^2+b^2+c^2) = 1.$$

Then we can conclude that $a^2 + b^2 + c^2 = 2$. So any point on the sphere of radius $\sqrt{2}$ has the desired property.

9.8.1. Let $x + y = uv$ and $xy = u - v$, and write x, y implicitly as functions of u, v. Calculate the possible derivatives.

Solution. Let $F(u, v, x, y) = (x + y - uv, xy - u + v)$. The total derivative is

$$DF = \begin{bmatrix} -v & -u & 1 & 1 \\ -1 & 1 & y & x \end{bmatrix}.$$

By the implicit function theorem we have that

$$\begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} = -\begin{bmatrix} 1 & 1 \\ y & x \end{bmatrix}^{-1} \begin{bmatrix} -v & -u \\ -1 & 1 \end{bmatrix} = \frac{1}{y - x} \begin{bmatrix} 1 - xv & -ux - 1 \\ vy - 1 & uy + 1 \end{bmatrix}.$$

This holds as long as $y - x \neq 0$.

9.8.2. Same as above except write x, v as functions of u, y.

Solution. Again by the implicit function theorem, we have

$$\begin{bmatrix} x_u & x_y \\ v_u & v_y \end{bmatrix} = \begin{bmatrix} -v & 1 \\ -1 & x \end{bmatrix}^{-1} \begin{bmatrix} 1 & -u \\ y & 1 \end{bmatrix} = \frac{1}{xv + y} \begin{bmatrix} x - y & -ux - 1 \\ 1 - vy & -u - v \end{bmatrix}.$$

This holds when $xv + y \neq 0$.

9.8.3. Two equations $F(x,y,u,v) = 0$ and $G(x,y,u,v) = 0$ determine x, y implicitly as functions of u, v. Find formulas for the derivatives.

Solution. As before, the implicit function theorem applies as follows.

$$\begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} = -\frac{1}{\frac{\partial(F,G)}{\partial(x,y)}} \begin{bmatrix} G_x & -F_y \\ -G_x & F_y \end{bmatrix} \begin{bmatrix} F_u & f_v \\ G_u & G_v \end{bmatrix} = -\frac{1}{\frac{\partial(F,G)}{\partial(x,y)}} \begin{bmatrix} \frac{\partial(F,G)}{\partial(u,y)} & \frac{\partial(F,G)}{\partial(v,y)} \\ \frac{\partial(F,G)}{\partial(u,x)} & \frac{\partial(F,G)}{\partial(v,x)} \end{bmatrix}$$

This holds when $\frac{\partial(F,G)}{\partial(x,y)}$ is nonzero.

9.8.4. The intersection of $2x^2 + 3y^2 - z^2 = 25$ and $x^2 + y^2 = z^2$ contains a curve C passing through $P = (\sqrt{7}, 3, 4)$. (a) Find a unit tangent vector T to C at P using the implicit function theorem. (b) Find one by determining a parametrization of C.

Solution. (a) We can write x, y as functions of z locally, and by the implicit function theorem

$$\begin{bmatrix} x_z \\ y_z \end{bmatrix} = -\begin{bmatrix} 4x & 6y \\ 2x & 2y \end{bmatrix}^{-1} \begin{bmatrix} -2z \\ -2z \end{bmatrix} = \begin{bmatrix} \frac{-2z}{x} \\ \frac{-z}{y} \end{bmatrix}.$$

Plugging in P, we obtain the unit tangent vector is

$$T = \frac{21}{\sqrt{5257}} \left(\frac{-8}{\sqrt{7}}, \frac{-4}{3}, 1 \right).$$

(b) On the other hand, we can write a parametrization as follows. By substituting for $2x^2$, we find that $y^2 + z^2 = 25$, which has parametrization $5(\cos(t), \sin(t))$. Then $x^2 = y^2 - z^2$, so that

$$\phi(t) = 5 \left(\sqrt{\cos^2(t) - \sin^2(t)}, \cos(t), \sin(t) \right).$$

Taking the derivative and plugging in P, we obtain the same vector.

9.8.5. Find a normal to the surface $F(u,v) = 0$ where $u = xy$ and $v = \sqrt{x^2 + z^2}$. Plug in some numbers when $(x,y,z) = (1,1,\sqrt{3})$, $D_uF(1,2) = 1$, and $D_vF(1,2) = 2$.

Solution. Let $f(x,y,z) = F(u,v)$ so that $\nabla(f)$ is the normal. Therefore it suffices to find Df, which by the chain rule is

$$Df = \begin{bmatrix} D_1F & D_2F \end{bmatrix} \begin{bmatrix} y & x & 0 \\ y/v & 0 & z/v \end{bmatrix} = \begin{bmatrix} yD_uF + \frac{x}{v}D_vF & xD_uF & \frac{z}{v}D_vF \end{bmatrix}.$$

Plugging in the numbers above, we obtain $n = (2, 1, \sqrt{3})$.

9.8.12. Let $F(x, y) = f(x + g(y))$ where $f, g : U \subseteq \mathbb{R} \to \mathbb{R}$. Find formulas for the first and second partials and verify $F_x F_{xy} = F_y F_{xx}$.

Solution. Let $T(x, y) = x + g(y)$, and $h : U \subseteq \mathbb{R} \to \mathbb{R}$ be any real valued function. Then $F_h = h \circ T$, so that by the chain rule

$$\begin{bmatrix} (F_h)_x \\ (F_h)_y \end{bmatrix} = \begin{bmatrix} h'(x + g(y)) \\ g'(y)h(x + g(y)) \end{bmatrix}.$$

Letting $h = f$, then we have the obvious formulae for F_x and F_y, and letting $h = f'$ and using the product rule, we obtain

$$D^2 F = \begin{bmatrix} f''(x + g(y)) & g'(y)f''(x + g(y)) \\ g'(y)f''(x + g(y)) & f''(x + g(y))g'(y)^2 + f'(x + g(y))g'(y) \end{bmatrix}.$$

Indeed

$$F_x F_{xy} = f'(x + g(y))g'(y)f''(x + g(y)) = F_y F_{xx}.$$

9.13.1,2,4,8,9. Classify the extrema of the following functions. (1) $z = x^2 + (y - 1)^2$ (2) $z = x^2 - (y - 1)^2$ (4) $z = (x - y + 1)^2$ (8) $z = x^2 y^3 (6 - x - y)$ (9) $z = x^3 + y^3 - 3xy$

Solution. (1) The critical points can be found by solving $\nabla(f) = (2x, 2y - 2) = (0, 0)$ which clearly has only solution $(0, 1)$. The Hessian at this point is $2I$, which is positive definite. Thus $(0, 1)$ is local minimum.

(2) Similarly, we need to solve $(2x, -2y + 2) = (0, 0)$ which still has solution $(0, 1)$. However, the Hessian matrix is

$$\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

which has negative determinant. Therefore $(0, 1)$ is a saddle point.

(4) To find the critical points, we solve $\nabla(f) = (2(x - y + 1), -2(x - y + 1)) = (0, 0)$. These equations are dependent, and so every point such that $y = x + 1$ is a critical point. The Hessian at a general point (x, y) is

$$\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

which is singular. Therefore the test is inconclusive. However, we see that each of this points is a minimum, for if $u = x - y + 1$, then the equation reads

$z = u^2$, which achieves minimum value exactly when $u = 0$, i.e. $y = x + 1$.

(8) To find the critical points, we solve

$$(2xy^3(6 - x - y) - x^2y^3, 3x^2y^2(6 - x - y) - x^2y^3) = (0, 0).$$

Pulling out a factor of xy^2, then we see that if $xy = 0$, then (x, y) is a critical point. If $xy \neq 0$, then we are left to solve

$$(2y(6-x-y)-xy, 3x(6-x-y)-xy) = (2(6-x-y)-x, 3(6-x-y)-y) = (0, 0).$$

Solving this linear system, we obtain $(x, y) = (2, 3)$.

For $(x, y) = (2, 3)$, then the Hessian implies that $(2, 3)$ is a local max.

For any point $xy = 0$, we must check manually since the Hessian has determinant 0 there. For $(x, y) = (0, 0)$, then (r, r) and $(-r, -r)$ have opposite sign values, so $(0, 0)$ is a saddle point. For $(x, 0)$ with $x \neq 6$, one can check that (x, ε) and $(x, -\varepsilon)$ have values with opposite signs so that in this case, (x, y) is a saddle point. Similarly, for $(0, 6)$, we see that $(\pm\varepsilon, 6)$ has opposite signed values, so $(0, 6)$ is a saddle point.

Now, at $(6, 0)$, we can write any line through this point (except $x = 6$, but that won't matter) as $y = mx - 6m$. Plugging this into z, we can rearrange the equation to say

$$z = -m(m + 1)x^2(6 - x)^4$$

which has only negative values around $(0, 6)$ for $m > 0$ and positive values for $m \in (-1, 0)$. Thus $(6, 0)$ is a saddle point.

For any point $(0, y)$ with $y < 0$ and $y > 6$, we see that ALL cross sections across $(0, y_0)$ is of the form $z = y_0^3 x^2(6 - y_0 - x)$, which has only nonpositive values in a neighborhood around $(0, y_0)$ in $\mathbb{R}^2$. Therefore, $(0, y)$ is a maximum in this range. For $(0, y)$ with $y \in (0, 6)$, then similarly the cross sections all have nonnegative values so $(0, y)$ is a maximum in this range.

(9) Taking the gradient and setting it equal to zero, we see the critical point satisfy $x^2 = y$ and $y^2 = x$. From this we obtain that $y^4 = y$, which is satisfies by only $0, 1$ in $\mathbb{R}$. If $y = 0$, then $x = 0$, and if $y = 1$ then $x = 1$. Therefore the critical points are $(0, 0)$ and $(1, 1)$.

The Hessian in general is $\begin{bmatrix} 6x & -3 \\ -3 & 6y \end{bmatrix}$, so at $(0,0)$, we have a saddle point and at $(1,1)$ we have a minimum.

9.13.21. Consider the least squares error function

$$E(a,b) = \sum_{i=1}^{n} (f(x_i) - y_i)$$

for a given set of points (x_i, y_i) and $f(x) = ax + b$. Find the (a,b) that achieves the minimum values of E.

Solution. We calculate the critical points of E as follows.

$$\nabla E(a,b) = \begin{bmatrix} \sum_i 2(ax_i + b - y_i)(x_i) \\ \sum_i 2(ax_i + b - y_i) \end{bmatrix} = \begin{bmatrix} a\left(\sum_i x_i^2\right) + b\left(\sum_i x_i\right) - \sum_i x_i y_i \\ a\left(\sum_i x_i\right) + nb - \sum_i y_i \end{bmatrix}$$

Setting the gradient equal to 0, then we must solve the linear system

$$\begin{bmatrix} \sum_i x_i^2 & \sum_i x_i \\ \sum_i x_i & n \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sum_i x_i y_i \\ \sum_i y_i \end{bmatrix}$$

which solves to

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{n \sum_i x_i^2 - \left(\sum_i x_i\right)^2} \begin{bmatrix} n \sum_i x_i y_i - \left(\sum_i x_i\right)\left(\sum_i y_i\right) \\ -\left(\sum_i x_i\right)\left(\sum_i x_i y_i\right) + \left(\sum_i x_i^2\right)\left(\sum_i y_i\right) \end{bmatrix}$$

To see that this point is a minimum, we note that the Hessian for all points is the matrix for the above system.

$$\begin{bmatrix} \sum_i x_i^2 & \sum_i x_i \\ \sum_i x_i & n \end{bmatrix}$$

The top left entry is positive since it is a magnitude of the vector $x = (x_1, \ldots, x_n)$. The determinant is positive by the Cauchy-Schwartz inequality.

$$\sum_i x_i \leq \sum_i |x_i| = (|x_1|, \ldots, |x_n|) \cdot (1, \ldots, 1) \leq \sqrt{n} \left(\sum_i |x_i|^2 \right)^{1/2}$$

Squaring the above inequality, shows the determinant is positive. Therefore the Hessian is positive definite, and the critical point is a minimum.

9.13.24. Let a be a stationary point of a scalar field f with continuous second-order partial derivatives in an n-ball $B(a)$. Prove that f has a saddle point at a if at least two of the diagonal entries of the Hessian matrix $H(a)$ have opposite signs.

Solution. By the second derivative test it suffices to show that two eigenvalues have opposite signs. Let $H(a) = [a_{ij}]$ and let λ_i be the eigenvalues. Since $H(a)$ is symmetric, it is diagonalizable by an orthogonal matrix $C = [c_{ij}]$.

Now, multiplying out $A = C\Lambda C^T$, we see that

$$a_{ii} = \lambda_1 c_{i1}^2 + \cdots + \lambda_n c_{in}^2.$$

Thus the diagonal entries are linear combinations of the eigenvalues with positive coefficients. Therefore if all λ_i have the same sign, then a_{ii} have the same sign. By contrapositive, if a_{ii} have opposite signs then so do two of the λ_i. This completes the proof.