Homework 7 Solutions
February 1, 2020

9.15.1. Find the extreme values of $z = xy$ with the constraint $x + y = 1$.

Solution. By Lagrange multipliers, we must solve

$$x + y = 1 \quad (y, x) = \lambda(1, 1).$$

Quickly, it is seen that $\lambda = x = y$, so that $2\lambda = 1$ and $\lambda = 1/2$. Therefore $(x, y) = (1/2, 1/2)$ and $z = 1/4$ is the extreme value.

9.15.5. Find the extreme values of the scalar field $f(x, y, z) = x - 2y + 2z$ on the sphere $x^2 + y^2 + z^2 = 1$.

Solution. By Lagrange multipliers, we must solve the system

$$x^2 + y^2 + z^2 = 1$$
$$(1, -2, 2) = \lambda(2x, 2y, 2z).$$

Solving for x, y, z in terms of λ, then plugging these expressions into g, we obtain $\lambda = \pm 3/2$. Therefore $x = \pm 1/3$, $y = \mp 2/3$, and $z = \pm 2/3$.

9.15.7. Find the shortest distance to the point $(1, 0)$ to the parabola $y^2 = 4x$.

Solution. The function that must be minimized is

$$f(x, y) = (x - 1)^2 + y^2$$

with the constraint $y^2 - 4x = 0$. By LM, then

$$(2x - 2, 2y) = \lambda(-4, 2y)$$

so that $2y(\lambda - 1) = 0$. If $\lambda = 1$, then $x = -1$ which is impossible due to the constraint, so therefore $y = 0$. In this case $x = 0$, so distance is minimized at the origin.

9.15.8. Find the points on the curve defined by $x^2 - xy + y^2 - z^2 = 1$ and $x^2 + y^2 = 1$ which are nearest to the origin.

Solution. By eliminating variables, we obtain the curve is on the surface $z^2 = xy$ with $x^2 + y^2 = 1$. Therefore we can parametrize the curve as

$$\gamma(t) = (\cos(t), \sin(t), \sqrt{\cos(t)\sin(t)})$$

for appropriate t Then we wish to minimize the function

$$||\gamma(t)||^2 = \cos(t)\sin(t) + 1.$$

Taking the derivative and setting equal to 0, we get that $t = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$. It is clear from plugging in that maximum distances or non well-defined points occur at these points. Therefore we check the boundaries at $t = n\pi/2$ for $n \in 0, 1, 2, 3$, and these yield minima.

9.15.13. Use Lagrange multipliers to find the extreme distances on the ellipse $x^2 + 4y^2 = 4$ to the line $x + y = 4$.

Solution. By the distance to a line formula, it suffices to minimize

$$f(x, y) = \frac{1}{2}(x + y - 4)^2$$

with respect to the constraint $g(x, y) = x^2 + 4y^2 - 4$. Lagrange multipliers implies that we can solve the system $g(x, y) = 0$ and $\nabla f = \lambda \nabla g$. The second equation gives us that

$$(x + y - 4, x + y - 4) = \lambda(2x, 8y)$$

so that $\lambda(x - 4y) = 0$. If $\lambda = 0$, then the whole ellipse is a solution, so throw away this possibility. Therefore $x = 4y$. Plugging into the ellipse equation, we obtain the two points $(x, y) = \pm(4, 1)/\sqrt{5}$. Therefore the minimum and maximum distances are $(4 + \sqrt{5})/\sqrt{2}$ and $(4 - \sqrt{5})/\sqrt{2}$.

10.5.2,7,8. Calculate a couple line integrals.

Solution. (2) In this problem, $F = (2a - y, x)$ and $\gamma(t) = a(t - \sin(t), 1 - \cos(t))$.

$$\begin{aligned}\int_\gamma F &= \int_0^{2\pi} (2a - a + a\cos(t), at - a\sin(t)) \cdot a(1 - \cos(t), \sin(t))\, dt \\ &= \int_0^{2\pi} a^2 t \sin(t)\, dt = -2\pi a^2\end{aligned}$$

(7) In this problem, $F = (x, y, xz - y)$ and $\gamma(t) = (1, 2, 4)t$ for $t \in [0, 1]$. Then by definition

$$\begin{aligned}\int_\gamma F &= \int_0^1 (t, 2t, 4t^2 - 2t) \cdot (1, 2, 4)\, dt \\ &= \int_0^1 16t^2 - 3t\, dt = 23/6.\end{aligned}$$

(8) In this problem $F = (x, y, xz - y)$ and $\gamma(t) = (t^2, 2t, 4t^3)$ for $t \in [0, 1]$.

$$\begin{aligned}\int_\gamma F &= \int_0^1 (t^2, 2t, 4t^5 - 2t) \cdot (2t, 2, 12t^2)\, dt \\ &= \int_0^1 48t^7 + 4t - 22t^3\, dt = 5/2\end{aligned}$$

10.5.10. Integrate the vector field $F = (x+y, x-y)/(x^2+y^2)$ on the circle of radius a centered at the origin traversed counterclockwise.

Solution. The parametrization is $\gamma(t) = a(\cos(t), \sin(t))$, and so by the line integral definition

$$\begin{aligned}\int_\gamma F &= \int_0^{2\pi} \frac{a}{a^2}(\cos(t)+\sin(t), \cos(t)-\sin(t)) \cdot a(-\sin(t), \cos(t))\,dt \\ &= \int_0^{2\pi} -\sin(t)\cos(t) - \sin^2(t) + \cos^2(t) - \sin(t)\cos(t) = 0\end{aligned}$$