Homework 8 Solutions
February 1, 2020

10.9.1. A force field F on $\mathbb{R}^3$ is $F(x,y,z) = (x, y, xz - y)$. Compute the work done by F on a particle moving from $(0,0,0)$ to $(1,2,4)$ along a straight line.

Solution. The parametrization of the curve γ is

$$\gamma(t) = (1,2,4)t.$$

Thus the integral is

$$\begin{aligned} W &= \int_\gamma F \\ &= \int_0^1 (t, 2t, 4t^2 - 2t) \cdot (1,2,4)\, dt \\ &= \int_0^1 t + 4t + 16t^2 - 8t\, dt = 23/6 \end{aligned}$$

10.9.2. Find the amount of work done by the force field $F(x,y) = (x^2 - y^2, 2xy)$ on a particle moving ccw once around the square bounded by the coordinate axes and the lines $x = a$ and $y = a$.

Solution. The parametrizations of each side are $\gamma_1(t) = (t, 0)$, $\gamma_2(t) = (a, t)$, $\gamma_3(t) = (a - t, a)$, and $\gamma_4(t) = (0, a - t)$. The work is calculated as follows.

$$\begin{aligned} W &= \sum_i \int_{\gamma_i} F \\ &= \int_0^a (t^2, 0) \cdot (1,0)\, dt + \int_0^a (a^2 - t^2, 2at) \cdot (0,1)\, dt \\ &+ \int_0^a ((a-t)^2 + a^2, 2(a-t)a) \cdot (-1,0)\, dt + \int_0^a (-(a-t)^2, 0) \cdot (0,-1)\, dt \\ &= 2a^3 \end{aligned}$$

10.9.5. Calculate the work done by the force field $F(x,y,z) = (y - z, z - x, x - y)$ along the curve of intersection of the sphere $r^2 = 4$ and the plane $z = y\tan(\theta)$ for fixed $\theta \in (0, \pi/2)$.

Solution. Let $a = \tan\theta$, since $\theta \in (0, \pi/2)$, then $a > 0$. Therefore we get a circle, but $z = ay$, so that

$$x^2 + y^2 + a^2y^2 = 4$$

which yields the ellipse
$$\frac{x^2}{2^2} + \frac{y^2}{\left(\frac{2}{\sqrt{1+a^2}}\right)^2} = 1.$$

We can rename $b = 2/(\sqrt{1+a^2})$, which is just $2/\sec\theta$. We note that b can range from $b = 2$ to $b = 0$. Then the parametrization is

$$\gamma(t) = (2\cos(t), b\sin(t), ab\sin(t)).$$

10.9.7. Calculate $\int_C (x+y)\,ds$ where C is the triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$, traversed ccw.

Solution. The parametrizations of the sides are $\gamma_1(t) = (t,0)$, then $\gamma_2(t) = (1-t,t)$, and $\gamma_3(t) = (0,t)$. Then the integral becomes

$$\begin{aligned}\int_C (x+y)\,ds &= \sum_i \int_{\gamma_i} (x+y)\,ds \\ &= \int_0^1 t\,dt + \int_0^1 1\sqrt{2}\,dt + \int_1^0 t\,dt \\ &= 1/2 + \sqrt{2} + 1/2 = \sqrt{2} + 1\end{aligned}$$

10.9.8. Calculate $\int_C y^2\,ds$, where C is the curve $a(t) = a(t - \sin(t), 1 - \cos(t))$.

Solution. The derivative is $a'(t) = a(1 - \cos(t), \sin(t))$, so that $||a'(t)|| = a\sqrt{2 - 2\cos(t)}$. Then the integral becomes

$$\int_C y^2\,ds = \int_0^{2\pi} a^3(1-\cos(t))^2\sqrt{2 - 2\cos(t)}\,dt.$$

10.9.11. Consider a uniform semicircular wire of radius a. (a) Show that the centroid lies on the axis of symmetry at a distance $2a/\pi$ from the center. (b) Show that the moment of inertia about the diameter through the end points of the wire is $Ma^2/2$, where M is the mass of the wire.

Solution. As usual,

$$\overline{x} = \frac{M}{L}\int_C x\,ds = \frac{M}{L}\int_0^\pi a\cos(t)(a)\,dt = 0$$

and

$$\overline{y} = \frac{M}{L}\int_C y\,ds = \frac{M}{a\pi}\int_0^\pi a\sin(t)(a)\,dt = 2a/\pi.$$

10.13.1acf. Determine whether the following sets are path connected or not, and explain how to construct a path between two points. (a) $S = \{(x, y) \mid x^2 + y^2 \geq 0\}$ (c) $S = \{(x, y) \mid x^2 + y^2 < 1\}$ (f) $S = \{(x, y) \mid x^2 + y^2 < 1 \text{ or } (x - 3)^2 + y^2 < 1\}$

Solution. (a) This set is connected. It is actually just all of $\mathbb{R}^2$. A straight line path between points will lie in $\mathbb{R}^2$.

(c) This an open ball of radius 1 centered at the origin. Similarly a straight line between any two points will also be in the ball.

(f) This set is a disjoint union of two open balls and is not connected. In fact there is no path between the point $(3, 0)$ and $(0, 0)$.

10.13.4. Given a vector field $F = (P, Q, R)$ on $\mathbb{R}^3$ where $F = \nabla\varphi$, prove some relations between the partials of P, Q, R.

Solution. Since $F = \nabla\varphi$ on some open S, then we have that $P = \varphi_x$, $Q = \varphi_y$, and $R = \varphi_z$. Then assuming F has continuous derivatives, we have that $P_y = \varphi_{xy} = Q_x$, $P_z = \varphi_{xz} = R_x$ and $Q_z = \varphi_{yz} = R_y$.

10.13.5. For each of the following vector fields, use the previous problem to show that F is not a gradient. Then find a closed path such that $\int_C F \neq 0$. (a) $F = (y, x, x)$ (b) $F = (xy, x^2 + 1, z^2)$.

Solution. (a) Since $P_z = 0 \neq 1 = R_x$, then by contraction of the result of 10.13.4, F cannot be a gradient. The path is

(b) Since $P_y = x$ and $Q_x = 2x$, then this contradicts the previous exercise, so F cannot be a gradient.

10.13.6. A force field F is defined on $\mathbb{R}^3$ by the equation $f(x, y, z) = (y, z, yz)$. (a) Determine whether or not f is conservative. (b) Calculate the work done in moving a particle on the curve described by $a(t) = (\cos(t), \sin(t), e^t)$ for $t \in [0, \pi]$.

Solution. (a) F is not conservative since $P_y = 1$ and $Q_x = 0$, which contradicts 10.13.4.

(b) By definition

$$\begin{aligned}\int_C F\cdot ds &= \int_0^\pi (\sin(t), e^t, e^t\sin(t))\cdot(-\sin(t), \cos(t), e^t)\,dt\\ &= \int_0^\pi -\sin^2(t) + e^t\cos(t) + e^{2t}\sin(t)\,dt\\ &= \frac{1}{10}(-3-5e^\pi+2e^{2\pi}-5\pi)\end{aligned}$$

10.13.9. A radial force F in the plane can be written $F(x,y) = f(r)\vec{r}$, where $\vec{r} = (x,y)$ and $r = ||\vec{r}||$. Show that such a force field is conservative.

Solution. Assume f is a differentiable function on $\mathbb{R}_{\geq 0}$. Then F is differentiable on all of $\mathbb{R}^2$ which is a simply connected domain. Thus we can decide that F is conservative by testing whether $F_{1y} = F_{2x}$. Indeed

$$F_{1y} = \frac{xy}{r}f'(r) = F_{2x}$$

so F is conservative.

In general if f is continuous, we can do the following. If we take the partials of some function differentiable function $G(r)$, we obtain $\nabla G = G'(r)/r(x,y)$. Lining up with F, we obtain that $f(r) = G'(r)/r$, so that $G(r) = \int rf(r)\,dr$ works as a potential.

10.18.1,3,6,9. Determine whether or not the following vector fields are conservative and if so find a potential. (1) $f = (x,y)$ (3) $f = (2xe^y + y, x^2e^y + x - 2y)$ (6) $f = (x,y,z)$ (9) $f = (3y^4z^2, 4x^3z^2, -3x^2yz^2)$

Solution. (1) The vector field f is conservative since a potential is $\varphi(x,y) = \frac{1}{2}(x^2+y^2)$.

(3) The vector field f is conservative since it is defined on a convex domain and the derivatives agree.

$$(f_1)_y = 2xe^y + 1 = (f_2)_x$$

We can find a potential by integrating each term with respect to the right variable and lining up all the terms. We obtain a potential which is $\varphi(x,y) = x^2e^y + xy - y^2$.

(6) The vector field is conservative since we can easily see a potential is $\varphi(x,y,z) = \frac{1}{2}(x^2+y^2+z^2)$.

(9) This vector field is not conservative since $P_y = 12y^3z^2$ and $Q_x = 12x^2z^2$, which are not equal. This contradicts a previous exercise.

10.18.13. A fluid flowing in $\mathbb{R}^2$ is given by a vector field where each particle is moving directly away from the origin. Assume if the particle is a distance r from the origin, its speed is ar^n. (a) Determine the values of a and n for which the velocity vector field is the gradient of some scalar field. (b) Find a potential function for the vector field whenever it is a gradient. Treat $n = -1$ separately.

10.18.14. If both φ and ψ are potential functions for a continuous vector field f on an open connected set S in $\mathbb{R}^n$, prove that $\varphi - \psi$ is constant on S.

Solution. Since both are potentials, then $\nabla(\varphi - \psi) = 0$. Since the derivative of the function $\varphi - \psi$ is 0, then we have already shown that $\varphi - \psi$ must be constant.

10.18.15. Let S be the set of all $x \neq 0$ in $\mathbb{R}^n$. Let $r = ||x||$, and let f be the vector field defined on S be $f(x) = r^p x$ where p is a real constant. Find a potential function for f on S. The case for $p = -2$ should be treated separately.

Solution. If $p = -2$, then we can see by inspection that $\varphi = \frac{1}{2}\ln(r^2)$ is a potential. In any other case, a potential is $\varphi = \frac{1}{2(p/2+1)}(r^2)^{p/2+1}$.