Homework 9 Solutions
February 1, 2020

11.9.1,3,5,6. Find some iterated integrals.

Solution. 1) $\int_0^1 \int_0^1 xy(x+y)\,dx\,dy = \int_0^1 y(3y+2)/6\,dy = 1/3$.

3) $\int_0^1 \int_1^3 \sqrt{y} + x - 3xy^2\,dy\,dx = -38/2 + 2\sqrt{3}$

5) $\int_0 \pi/2 \int_0^{\pi/2} \sin(x+y)\,dx\,dy = 2$

6) Let P be the region where $\cos(x+y)$ is positive, i.e. when $x+y < \pi/2$ and $x+y > 3\pi/2$. Let N be the other region, where $\cos(x+y)$ is negative. Then

$$\int_0^\pi \int_0^\pi |\cos(x+y)|\,dx\,dy = \int\int_P \cos(x+y)\,dx\,dy - \int\int_N \cos(x+y) = 2\pi.$$

11.9.10. Let f be defined on the rectangle $Q = [0,1] \times [0,1]$ as

$$f(x,y) = \begin{cases} 1 - x - y & \text{if } x + y \leq 1 \\ 0 & \text{else} \end{cases}.$$

Make a sketch of the ordinate set over Q, and compute the volume.

Solution. The ordinate set looks like wedge, with zero value on the top right triangle of Q. The volume is $\int\int_Q f(x,y)$, which we can compute as follows.

$$\int\int_Q f(x,y) = \int_0^1 \int_0^{1-x} 1 - x - y\,dy\,dx = 1/6.$$

Note that the inside integral as a function of x is exactly $\int_0^{1-x} 1 - x - y\,dy$.

11.9.14. Let f be defined on $Q = [0,1]^2$ by

$$f(x,y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}.$$

Prove that $\int\int_Q f$ exists and is 0.

Solution. We show that $\underline{I} = \overline{I} = 0$. Let s be a step function on Q such that $s \leq f$. We claim that $s(x) \leq 0$. If $\{Q_i\}_{i \in I}$ is the subdivision of Q for s, let $C \subset I$ be the subset of indices such that $Q_i \cap \Delta \neq \emptyset$, where Δ is the diagonal. For $i \notin C$, then $f = 0$ on Q_i and so $s(x) \leq 0$ by defintion. For $i \in C$, then every rectangle containing a point of Δ will also contain a point off of Δ (add some small amount to the x or y value). Therefore $s(x) \leq 0$ on Q_i in this case as well. This proves that 0 is an upper bound for $\{\int\int_Q s\}$. In fact it is the least upper bound, since $s = -\varepsilon$ for all $\varepsilon > 0$ satisfies $s \leq f$ and $\int\int_Q -\varepsilon = -\varepsilon$. We conclude that $\underline{I} = 0$.

Similarly, a symmetric argument shows that for all step functions $t \geq f$, $t = 1$ for Q_i that intersect the diagonal and 0 else. Since $t \geq f$, we know that 0 is a lower bound for the integrals of t, but we show it is the greatest lower bound, but constructing a $t \geq f$ with $\int\int_Q t = 1/n$.

In particular, define

$$Q_{ij} = \left[\frac{i-1}{n}, \frac{i}{n}\right] \times \left[\frac{i-1}{n}, \frac{i}{n}\right]$$

for $1 \leq i, j \leq n$. This forms a subdivision of Q. Define $t(x, y)$ as follows.

$$t(x,y) = \begin{cases} 1 & \text{if } (x,y) \in Q_{jj} \\ 0 & \text{else} \end{cases}$$

Then

$$\int\int_Q t(x,y) = \sum_j \int\int_{Q_{jj}} 1 = n\frac{1}{n^2} = \frac{1}{n}.$$

Thus the greatest lower bound possible is 0, and $\overline{I} = 0$. This completes the proof.

1a.. Give an example of a bounded function $f : Q = [0,1]^2 \to \mathbb{R}$ which is not integrable.

Solution. Let

$$f(x,y) = \begin{cases} 1 & (x,y) \in \mathbb{Q} \times \mathbb{Q} \\ 0 & \text{else} \end{cases}.$$

Then f is not integrable because the lower integral $\underline{I} = \sup_{s \leq f} \int\int_Q s = 0$. Every partition of Q, there always exists a point with irrational component in each smaller rectangle. Thus $s(x,y) \leq 0$ for all x, y. Similarly, there is a rational point in every rectangle of every partition of Q, so that $\overline{I} = \inf_{t \geq f} \int\int_Q t = 1$. Thus $\underline{I} < \overline{I}$ and the integral doesn't exist.

1b.. Give an example of a function $g : [0,1]^2 \to \mathbb{R}$ so that for all fixed x, the integral $\int_0^1 g(x,y)\,dy$ exists but for some fixed y, $\int_0^1 g(x,y)\,dx$ does not exist.

Solution. Let $g(x,y) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & \text{else} \end{cases}$. Since x is fixed, we have that

$$\int_0^1 g(x,y)\,dy = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & \text{else} \end{cases}.$$

But for any y, say $y = 0$, we have $\int_0^1 g(x,y)\,dx$ does not exist, similarly to 1a.

1c. Give an example of a bounded function $h : [0,1]^2 \to \mathbb{R}$ so that for all fixed x, $\int_0^1 h(x,y)\,dy$ exists, but the iterated integral does not exist.

Solution. We let $h(x,y) = g(x,y)$ from the previous problem. Then $A(x) = \int_0^1 h(x,y)\,dy = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & \text{else} \end{cases}$, and we saw in 1a that $\int_0^1 A(x)\,dx$ does not exist.