Midterm 3 Review
April 17, 2018

1. Let $x = u^2 - v^2$ and $y = 2uv$ and let $u^2 + v^2 \leq 1$ and $u, v \geq 0$. Find the x and y range D and evaluate $\iint_D 1\, dx\, dy$.

Solution. First, we find the x, y range D. The region for (u, v) coordinates is a quarter of a circle in the first quadrant. For $(u, v) = (u, 0)$, then $(x, y) = (u^2, 0)$. For $(u, v) = (0, v)$, then $(x, y) = (-v^2, 0)$. Thus the two lines of the quarter circle get sent to a horizontal line segment on the x-axis from -1 to 1. Now if (u, v) lies on the circle part, we let $(u, v) = (\cos t, \sin t)$ from $t = 0$ to $t = \pi/2$. Then $(x, y) = (\cos^2 t - \sin^2 t, 2\cos t \sin t) = (\cos 2t, \sin 2t)$. Then we see that the quarter circle part gets sent a half circle part! Thus D is the upper half of a circle. Thus $\iint_D 1\, dx\, dy = \pi/2$.

2. Find the volume of a cone with base radius r and height h using a triple integral.

Solution. If C is the cone and D is the circle of radius r in the xy-plane then

$$\iiint_C 1\, dV = \iint_D \int_0^{h - \frac{h}{r}\sqrt{x^2+y^2}} 1\, dz\, dx\, dy = \int_0^{2\pi} \int_0^r \left(h - \frac{h}{r}\rho\right)(\rho)\, d\rho\, d\theta = \frac{1}{3}\pi r^2 h$$

3. Calculate $\iint_S \nabla \times F \cdot dS$ where $F(x, y, z) = (x^3, -y^3, 0)$ and S is the hemisphere $x^2 + y^2 + z^2 = 1$ and $x \geq 0$.

Solution. By Stokes' theorem,

$$\iint_S \nabla \times F \cdot dS = \int_{\partial S} F \cdot ds$$

so we need to parametrize the unit circle in the yz-plane. This is $c(t) = (0, \cos t, \sin t)$ with the wrong orientation, so we introduce a negative sign. The integral is

$$-\int_{\partial S} F \cdot ds = -\int_0^{2\pi} (0, 1 - \cos^3 t, 0) \cdot (0, -\sin t, \cos t)\, dt = \int_0^{2\pi} -\sin t \cos^3 t\, dt = 0.$$

4. Let S be the triangle with corners $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ with outward normal. Let $F(x, y, z) = (yz, xz, xy)$. Find $\iint_S F \cdot dS$.

Solution. Parametrize S by $(x, y, 1-x-y)$ where $x, y \geq 0$ and $x+y \leq 1$. The normal is $(1,1,1)$ constantly, so

$$\iint_S F \cdot dS = \int_0^1 \int_0^x y(1-x-y) + x(1-x-y) + xy \, dy \, dx = 1/8.$$

Here's the hard way, which I for some reason did first. Don't do it this way. Notice that $F = \nabla \times H$ where $H(x,y,z) = (\frac{x}{4}(z^2-y^2), \frac{y}{4}(x^2-z^2), \frac{z}{4}(y^2-x^2))$. Then let c_1, c_2, and c_3 be the sides of the triangle. So that

$$\iint_S F \cdot dS = \int_{c1+c2+c3} H \cdot ds$$

Doing all of these integrals you get the answer is 1/8.

5. Let S be the surface $z = xy + 1$ graphed on the top right quarter of the unit disc with boundary ∂S. Let $F(x,y,z) = (x, 2z, y)$. Find

$$\int_{\partial S} F \cdot ds.$$

Solution. By Stokes' theorem, we can find the integral of $\nabla \times F$ on S with outward normal. Indeed the parametrization

$$\Phi(r,t) = (r \cos t, r \sin t, r^2 \cos t \sin t + 1)$$

for $0 \leq r \leq 1$ and $0 \leq t \leq \pi/2$ is outward pointing. The normal vector is

$$\begin{aligned} n = \frac{\partial \Phi}{\partial r} \times \frac{\partial \Phi}{\partial t} &= (\cos t, \sin t, r * \sin 2t) \times (-r \sin t, r \cos t, r^2 \cos 2t) \\ &= (-r^2 \sin t, -r^2 \cos t, r) \end{aligned}$$

Also $\nabla \times F = (-1, 0, 0)$. So the integral is

$$\begin{aligned} \int_{\partial S} F \cdot ds &= \iint_S \nabla \times F \cdot dS \\ &= \int_0^{\pi/2} \int_0^1 (-1,0,0) \cdot (-r^2 \sin t, -r^2 \cos t, r) \, dr \, dt \\ &= \left(\int_0^{\pi/2} \sin t \, dt \right) \left(\int_0^1 r^2 \, dr \right) = 1/3 \end{aligned}$$

6. Find the area of the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ using Green's theorem.

Solution. Let D be the filled in ellipse and E be the boundary ellipse. By Greens theorem

$$\iint_D 1\,dA = \frac{1}{2}\int_E y\,dx - x\,dy = \frac{1}{2}\int_0^{2\pi} ab\cos^2 t + ab\sin^2 t\,dt = \pi ab.$$

7. Let S be the surface parametrized by $\Phi(u,v) = (2\sin(u), 3\cos(u), v)$ for $0 \leq u \leq 2\pi$ and $0 \leq v \leq 1$. Let $F(x,y,z) = (x,y,z)$. Find (or at least set up) $\iint_S F \cdot dS$.

Solution. The integral is

$$\int_0^1 \int_0^{2\pi} (2\sin(u), 3\cos(u), v)\cdot(-3\sin u, -2\cos u, 0)\,du\,dv = \int_0^{2\pi} -6\,du = -12\pi.$$

8. Find the surface area of a sphere with radius r.

Solution.

$$\int_0^{2\pi}\int_0^{\pi} |T_\theta \times T_\varphi|\,d\theta\,d\varphi = \int_0^{2\pi}\int_0^{\pi} r^2 \sin\varphi\,d\theta\,d\varphi = 4\pi r^2.$$

9. Let $F(x,y,z) = (x,y,z)$ and let S be the sphere. Find $\iint F \cdot dS$. (Hint: is there a shortcut?)

Solution. I believe the normal is $n = \sin\varphi(x,y,z) = \sin\varphi F$ so the integral becomes

$$\int_0^{2\pi}\int_0^{\pi} F\cdot(\sin\varphi)F\,d\varphi\,d\theta = \iint \sin\varphi |F|^2\,d\varphi\,d\theta = \iint \sin\varphi\,d\varphi\,d\theta = 4\pi.$$

10. Let $F = (2xz, 1, x^2)$. Let c be the contour of straight lines that follow the points

$$(0,0,0) \to (1,0,0) \to (1,1,0) \to (1,1,1) \to (0,1,1) \to (0,0,1) \to (0,0,0).$$

Find $\int_C F \cdot ds$. What is the arclength of c?

Solution. We see that $F = \nabla f$ where $f(x,y,z) = x^2 z + y$. Since c is a closed the fundamental theorem of calculus says that $\int F \cdot ds = 0$. The arclength is 6.

11. Find the surface area of the region inside $x^2+z^2 = 2$ and $x^2+y^2+z^2 = 4$.

Solution. First we have to find where the cylinder intersects the sphere. Plugging $x^2 + z^2 = 2$ into the sphere equation, we get $y^2 + 2 = 4$ so $y = \pm\sqrt{2}$ as our answer. This is the y value where the intersections occur. Thus the length of the cylinder is $2\sqrt{2}$ and the radius is $\sqrt{2}$ so the surface area of the cylinder part is $2\pi rh = 8\pi$.

For the sphere part, we have to find the angle where these intersections occur. We can also just assume this is happening vertically since its symmetric. Making a cross section of this shape for when, we notice that the cylinder hits the sphere at $(y, z) = (\sqrt{2}, \sqrt{2})$ on a sphere of radius 4. Then $\varphi = \arctan(\sqrt{2}/\sqrt{2}) = \pi/4$. Thus our ending angle of the integral is $\pi/4$. So the surface area of one the spherical caps is

$$\int_0^{2\pi} \int_0^{\pi/4} \sin\varphi \, d\varphi \, d\theta = 2\pi(1 - \sqrt{2}/2).$$

In total we get $\pi(8 + 4 - 2\sqrt{2})$.

12. Set up the line integral of the function $f(x, y, z) = xyz$ around the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x = z$.

Solution. The parametrization of the curve is $c(t) = (\cos t, \sin t, \cos t)$. Then the integral is

$$\int_0^{2\pi} \cos^2 t \sin t \sqrt{2\sin^2 t + \cos^2 t} \, dt.$$

13. Find the volume of the region outside of the cone $x = 3\sqrt{y^2 + z^2}$ and inside the sphere $x^2 + y^2 + z^2 = 4$.

Solution. Making everything vertical, the angle from the z-axis to the cone is $\arctan 1/3$ so volume is

$$V = \int_0^2 \int_0^{2\pi} \int_{\arctan 1/3}^{\pi} \rho^2 \sin\varphi \, d\varphi \, d\theta \, d\rho = 2\pi \frac{8}{3} \left(1 + \frac{3}{\sqrt{10}}\right).$$

14. Let $s = 3x + 2y$ and $t = x - y$. Find the integral $\iint_D 3x^2 - xy - 2y^2 \, dx \, dy$ where D is parallelogram $(0, 0)$, $(1, 1)$, $(2, -3)$, and $(3, -2)$ by change of variables.

Solution. The region in s, t coordinates is $0 \le s \le 5$ and $0 \le t \le 5$. Then $x = s/5 + 2t/5$ and $y = s/5 - 3t/5$. The Jacobian $|J| = 1/5$ so the integral becomes

$$\iint_D (3x + 2y)(x - y) \, dx \, dy = \frac{1}{5} \int_0^5 \int_0^5 st \, ds \, dt = 125/4.$$

15. Let $F(x,y,z) = (x,y,z)$. Is there a vector field G such that $\nabla \times G = F$. What if $F(x,y,z) = (x,y,-2z)$?

Solution. No. Yes. Take the div.

16. Let $F = (x^2+y^2, x^2-y^2)$ and let c be the straight line from $(2,1)$ to $(3,4)$. Find the line integral $\int_c F \cdot ds$.

Solution. The parametrization is $l(t) = (2+t, 1+3t)$ with $t=0$ to $t=1$. So the integral is $\int_0^1 F(l(t)) \cdot (1,3)\, dt = 34/3$.

17. Let C be the cylinder $1 = x^2+z^2$ from $-1/2 \le y \le 1$ with outward normal vector and let

$$F(x,y,z) = (xy, 0, -yz).$$

Find $\iint_S F \cdot dS$ using Stokes' theorem.

Solution. Since $\nabla \cdot F = 0$ then there is a G such that $\nabla \times G = F$. By Stokes' we can integrate G around the two edges of the cylinder, each going the opposite way. But then we can reapply Stokes' theorem and integrate F on any surface whose boundary is the two circles! We can pick two disks of radius one. The disk D_1 at $y=-1/2$ has normal $n_1 = (1,0,0)$ facing right, while D_2 at $y=1$ has normal $n_2 = (-1,0,0)$ facing left.

So

$$\iint_S F \cdot dS = \iint_{D_1} F \cdot dS + \iint_{D_2} F \cdot dS = \iint_D xz\, dA + \iint_D -xz\, dA = 0.$$