Math 4242
Fall 2020
Practice Final Exam
12/21/2020
Time Limit: 120 minutes

Name (Print): ____________________

Instructor ____________________

Exam 1 contains 4 pages (including this cover page) and 9 problems. Please check to see if any pages are missing.

Work individually without reference to a textbook, the internet, or a calculator.
However, hand written notes are permitted.
Show your work on each problem. Specifically:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Unsupported answers will not receive credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations will likely receive partial credit.
- **Circle your final answer** for problems involving a series of computations.

Please do not write in the table to the right.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	30	
6	20	
7	30	
8	20	
9	20	
Total:	200	

1. (20 points) Consider the matrix

$$A = \begin{pmatrix} 2 & 3 & -2 \\ -3 & 0 & 0 \\ -4 & -6 & 4 \end{pmatrix}.$$

   (a) (5 points) Find the eigenvalues of the matrix A.

   (b) (5 points) Find the algebraic and geometric multiplicities of each eigenvalue from part (a).

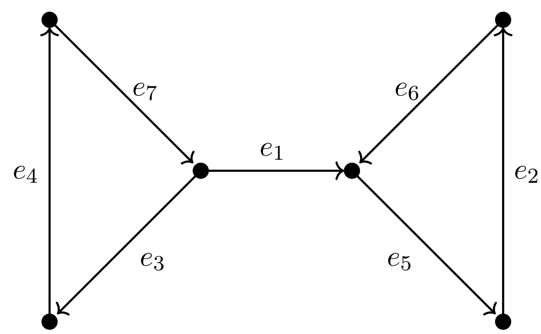
   (c) (10 points) Compute the Jordan decomposition of A.

2. (20 points) Consider the following linear system of equations.

$$\begin{aligned} 2x + 3y &= 0 \\ -x - y &= 2 \\ 2x + y &= 0 \end{aligned}$$

   Compute the least squares solution $\vec{x}^*$ to the system.

3. (20 points) Consider the following connected digraph with the edges labeled.



   (a) (10 points) Use the vertex-edge formula to determine the number of independent circuits of the graph. Explain why your answer makes sense visually.

   (b) (10 points) Use the boundary operator to determine whether the linear combination

$$e_1 + e_2 - 2e_3 - 2e_4 + e_5 + e_6 - 2e_7$$

   represents a circuit in the graph or not.

4. (20 points) Consider the formula

$$\langle (x_1, y_1), (x_2, y_2) \rangle = 2x_1x_2 - x_1y_2 - x_2y_1 + y_1y_2,$$

   (a) (15 points) Show that the above $\langle -, - \rangle$ defines an inner product by proving it satisfies the three properties of an inner product. (Hint: Use your knowledge of eigenvalues for one of the properties.)

   (b) (5 points) Write out the Cauchy-Schwartz inequality

$$|\langle v, w \rangle| \leq \|v\| \, \|w\|$$

   explicitly for this inner product in particular, where $v = (v_1, v_2)$ and $w = (w_1, w_2)$.

5. (30 points) Consider a 4×4 matrix M with columns v_1, v_2, v_3, v_4. Suppose that the RREF of M is

$$\begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

   (a) (10 points) Determine whether v_1, v_2, v_3, v_4 form a basis of any $\mathbb{R}^n$. Justify your answer in terms of linear independence and span.

   (b) (10 points) Compute $\ker(M)$ and $\dim(\ker(M))$.

   (c) (5 points) Find the rank of M.

   (d) (5 points) Is M invertible? Explain your answer.

6. (20 points) Let $Q_1 = \{(x, y) \mid x, y > 0\}$ be the first quadrant of the xy-plane. We give Q_1 a vector space structure by defining $(x, y) + (u, v) = (xu, yv)$ and $c(x, y) = (x^c, y^c)$. Recall that the zero element is $\vec{0} = (1, 1)$ and $-(x, y) = (x^{-1}, y^{-1})$ in this case. You need not show that this is a vector space

   Define a function $\exp : \mathbb{R}^2 \to Q_1$ by the formula

$$\exp(x, y) = (e^x, e^y).$$

   Show that exp is a linear function by proving it satisfies the two properties of being a linear function.

7. (30 points) Consider the basis of $\mathbb{R}^3$

$$\beta = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

   (a) (15 points) Use the Gram-Schmidt algorithm to turn β into an orthonormal basis.

   (b) (15 points)Use your answer from part (a) to compute the projection $\text{proj}_W(v)$ where $v = (2, 1, 2)$ and W is the span of the first two original basis vectors, i.e.

$$W = \text{span}\left(\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \right).$$

8. (20 points) Let

$$A = \begin{pmatrix} 0 & 0 & -3 \\ 2 & 2 & -1 \\ 1 & 1 & -2 \end{pmatrix}.$$

   (a) (10 points) Compute A^3.

   (b) (10 points) Compute e^A.

9. (20 points) Let A be a 3×3 diagonalizable matrix, with decomposition $A = S\Lambda S^{-1}$, where Λ is the diagonal matrix of eigenvalues and S is the change of basis matrix. Let $p_A(\lambda) = \det(A - \lambda I)$ denote the characteristic polynomial

$$p_A(\lambda) = c_3\lambda^3 + c_2\lambda^2 + c_1\lambda + c_0.$$

   We can also denote $p_A(M)$ to the matrix version of the characteristic polynomial

$$p_A(M) = c_3M^3 + c_2M^2 + c_1M + c_0I.$$

   (a) (10 points) Prove that $p_A(\Lambda) = 0$, where 0 is the zero matrix.

   (b) (10 points) Use the diagonal decomposition to prove that $p_A(A) = 0$, i.e. a 3×3 diagonalizable matrix satisfies its own characteristic polynomial.

   (Hint: Don't write out the matrix A entry by entry! Use what you know about matrix algebra and the roots of the characteristic polynomial!)

   (Note: Actually all $n \times n$ matrices satisfy their own characteristic polynomial. This is called the Cayley-Hamilton Theorem.)